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An Elliptic Neumann Problem Involving Critical Exponent

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

This paper is devoted to study the following nonlinear elliptic problem with Neumann boundary condition, $(P_\mu)\,:\, -\Delta u + \mu u = K u^3$, $u > 0$ in Ω and $\partial u / \partial \nu = 0$ on $\partial\Omega$ where Ω is a smooth bounded domain in $\mathbb{R}^4$, μ is a positive parameter and K is a C^3 positive Morse function on $\overline{\Omega}$. Using dynamical methods involving the study of *Palais-Smale condition* of the associated variational structure J, we prove some existence results of (P_μ).

Keywords: Variational problem; critical points; palais- smale condition.

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1 Introduction

Let us consider the nonlinear Neumann elliptic problem:

$$(P_{q,\mu}) \quad \begin{cases} -\Delta u + \mu u &= \quad u^q, \quad u > 0 \qquad \text{in } \Omega \\ \quad \frac{\partial u}{\partial \nu} &= \quad 0 \qquad\qquad\quad \text{on } \partial\Omega, \end{cases}$$

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where $1 < q < \infty$, Ω is a smooth bounded domain in $\mathbb{R}^4$, μ is a positive parameter and $\frac{\partial u}{\partial \nu}$ is the normal derivative of u.

It is well known that problem $(P_{q,\mu})$ appears in several domains of applied sciences. For example, in biological pattern formation, it was used as a steady-state equation for the shadow system of the Gierer- Meinhardt system [1] and as parabolic equations in chemotaxis, (Keller-Segel model [2]).

For the subcritical case, i.e.$q < \frac{n+2}{n-2}$, it was proved by Lin, Ni and Takagi [2] that, if μ is very small, the only solution of this problem is the constant one, however they proved that this problem has a nonconstant solutions which blow up at one or several points for large μ. concerning the critical case, i.e. $q = 5$, it was shown that, the only solution of problem $(P_{q,\mu})$ is the constant one when μ is small and in convex domains [3].

Note that, many works has been devoted to study the solutions of problems of type $(P_{q,\mu})$ with the Dirichlet boundary conditions see for example [4], [5], [6], [7], [8], [9], [10].

In this paper, we study problem $(P_{q,\mu})$ for fixed μ when $n = 4$ and the exponent $q = 3$ is critical:

$$(P_\mu) \quad \begin{cases} -\Delta u + \mu u &= Ku^3, \quad u > 0 \qquad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} &= 0 \qquad\qquad\qquad \text{on } \partial\Omega, \end{cases}$$

where K is a C^3 positive Morse function on $\overline{\Omega}$.

Our goal is to to provide some sufficient conditions of the function K under which the problem (P_μ) has a positive solution.

Before stating the theorems, we will introduce the following notations and assumptions.
For $a \in \overline{\Omega}$ and $\lambda > 0$, let

$$\delta_{(a,\lambda)}(x) = c_0 \frac{\lambda}{1 + \lambda^2 |a - x|^2}$$

where c_0 is chosen so that $\delta_{(a,\lambda)}$ is the family of solutions of the following problem

$$-\Delta u = u^3, \quad u > 0, \quad \text{in } \mathbb{R}^4.$$

$(\mathbf{H_1})$ Let y_0 be a maximum of the function $K_1 = K_{|\partial\Omega}$ and $\max K(y)_{y\in\Omega} \le 2K_1(y_0)$.
$(\mathbf{H_2})$ $c_3 \frac{\partial K}{\partial \nu}(y_0) - \frac{4}{3}\pi w_2 K_1(y_0)\mathcal{H}(y_0) < 0$ where c_3, w_2 are constants defined bellow.

In the assumption $(\mathbf{H_2})$, we also denote by $\mathcal{H}$ for the mean curvature of the boundary of Ω .
In the first part of this work, we establish the following existence result.

Theorem 1.1. *Suppose that the function K satisfy the assumptions (H_1) and (H_2). Then problem (P_μ) has a solution under the level $c_\infty := (S_4/2)^{1/2} K_1(y_0)^{-1/2}$.*

The proof of this theorem is based on the fact that the associate functional J does not satisfy the *Palais-Smale condition* along the flow lines under the level c_∞ defined at the point y_0 which is in the boundary. The same argument can be applied if the level c_∞ defined at an interior point. This is our aim in the second part of this paper.

$(\mathbf{H_3})$ Assume that y_1 is a maximum of the function K in Ω and

$$\max K(y)_{y\in\partial\Omega} \le K(y_1)/2.$$

We have the following result

Theorem 1.2. *We suppose that the assumption (H_3) holds, the problem (P_μ) has a solution under the level $d_\infty := (S_4)^{1/2} K(y_1)^{-1/2}$.*

To briefly outline the remainder of the paper, we introduce the variational function associate to the problem (P_μ) and present a basic preliminaries in Section 2. In Section 3, we give some careful expansions of J associated to the problem (P_μ). The proofs of theorems will be carried out in Section 4.

2 Preliminary Results

Let us define the following variational formulation corresponding to the problem (P_μ) :

$$J(u) = \frac{\int_\Omega |\nabla u|^2 + \mu \int_\Omega u^2}{\left(\int_\Omega K|u|^4 \right)^{1/2}}, \qquad u \in H^1(\Omega). \tag{2.1}$$

It is well known that the critical points of this variational formulation J are solutions of problem (P_μ) up to constant multipliers. In the sequel, we will assume that the space $H^1(\Omega)$ is equipped with the norm $\|.\|$ and its corresponding inner product $\langle .,. \rangle$ defined by

$$\|w\|^2 = \int_\Omega |\nabla w|^2 + \mu \int_\Omega w^2, \quad \text{and} \quad \langle w, v\rangle = \int_\Omega \nabla w \nabla v + \mu \int_\Omega wv, \quad w, v \in H^1(\Omega)$$

We set $\Sigma = \{u \in H^1(\Omega) \,/\, \|u\|^2 = 1\}$ and $\Sigma^+ = \{u \in \Sigma \,/\, u \geq 0\}$.

Note that the functional J defined by (2.1) does not satisfy the Palais-Smale condition on Σ^+. Many authors have studied the failure of The Palais-Smale condition for J (see Brezis-Coron [11], Lions [12], Rey [13], Struwe [14]).

In the following we will describe the sequences that fail the Palais-Smale condition for J.
For $\varepsilon > 0$ small enough and $p \in \mathbb{N}^*$, we define

$$V(p,\varepsilon) = \Big\{ u \in \Sigma^+ / \exists\, a_1, ..., a_p \in \overline{\Omega}, \exists\, \lambda_1, ..., \lambda_p > 0, \text{ s.t. } ||u - \sum_{i=1}^{p} K(a_i)^{-\frac{1}{2}} \delta_i|| < \varepsilon,$$
$$\lambda_i > \varepsilon^{-1}, \varepsilon_{ij} < \varepsilon \text{ and } \lambda_i d_i < \varepsilon \text{ or } \lambda_i d_i > \varepsilon^{-1} \Big\}$$

where $\delta_i = \delta_{(a_i,\lambda_i)}$, $d_i = d(a_i, \partial\Omega)$ and $\varepsilon_{ij}^{-1} = \lambda_i/\lambda_j + \lambda_j/\lambda_i + \lambda_i\lambda_j|a_i - a_j|^2$.

Proposition 2.1. *(see [15], [12] and [16]) We suppose that there is no critical point of J in Σ^+ and let $(u_r) \in \Sigma^+$ be a sequence such that $J(u_r)$ is bounded and $\nabla J(u_r) \to 0$. Then, there exist an extracted subsequence of u_r, denoted also (u_r), a sequence $\varepsilon_r > 0$ $(\varepsilon_r \to 0)$ and an integer $p \in \mathbb{N}^*$ such that $u_k \in V(p, \varepsilon_k)$.*

For sake of simplicity, we will suppose, in the sequel, that If $u \in V(p, \varepsilon)$, then

$$\lambda_i d_i < \varepsilon \text{ when } i \leq q \text{ and for } i > q, \quad \varepsilon^{-1} < \lambda_i d_i.$$

3 Some Useful Estimations

In this section, we will study the Euler functional J associated to problem (P_μ). We will determine some expansions of J which are useful in the proof of our results.

Proposition 3.1. *For $\varepsilon > 0$ small enough and $u = \sum_{i=1}^{p} K(a_i)^{-\frac{1}{2}}\delta_{(a_i,\lambda_i)} \in V(p,\varepsilon)$, we have the following expansion*

$$J(u) = (\frac{S_4}{2})^{1/2}\Big(\sum_{i=1}^{q} K(a_i)^{-1} + 2\sum_{i=q+1}^{p} K(a_i)^{-1}\Big)^{1/2}\Big[1 + \frac{c_3}{\theta}\sum_{i\le q}\frac{1}{\lambda_i K(a_i)^2}\frac{\partial K}{\partial \nu}(a_i) - \frac{4\pi w_2}{3\theta}\sum_{i\le q}\frac{\mathcal{H}(a_i)}{\lambda_i K(a_i)}$$
$$+ O\Big(\sum_{r\neq k}\varepsilon_{kr} + \sum_{i>q}\frac{1}{(\lambda_i d_i)^3} + \mu\sum_i \frac{\log\lambda_i}{\lambda_i^2}\Big)\Big],$$

where

$$\theta = \frac{S_4}{2}\Big(\sum_{i=1}^{q} K(a_i)^{-1} + 2\sum_{i=q+1}^{p} K(a_i)^{-1}\Big) \quad ; \qquad S_4 = \int_{\mathbb{R}^4}\delta_{(0,1)}^4 \quad ;$$
$$w_2 = \int_{\mathbb{R}^4}\delta_{(0,1)}^4 y^2 \quad ; \qquad c_3 = \int_{\mathbb{R}^4}\frac{x_4 dx}{(1+|x|^2)^4}$$

Proof. We have

$$J(u) = \frac{\int_\Omega |\nabla u|^2 + \mu\int_\Omega u^2}{\Big(\int_\Omega K|u|^4\Big)^{1/2}} = \frac{N}{D^{1/2}}, \qquad u \in H^1(\Omega). \tag{3.1}$$

$$\int_\Omega |\nabla u|^2 = \sum_i \int_\Omega K(a_i)^{-1}|\nabla\delta_i|^2 + \sum_{i\neq j}\int_\Omega K(a_i)^{-1/2}K(a_j)^{-1/2}\nabla\delta_i\nabla\delta_j \tag{3.2}$$

$$\int_\Omega u^2 = \int_\Omega (\sum_i K(a_i)^{-1/2}\delta_i)^2 = \sum_i\int_\Omega K(a_i)^{-1}\delta_i^2 + \sum_{i\neq j}\int_\Omega K(a_i)^{-1/2}K(a_j)^{-1/2}\delta_i\delta_j \tag{3.3}$$

So

$$N = \sum_i\int_\Omega K(a_i)^{-1}|\nabla\delta_i|^2 + \mu\sum_i\int_\Omega K(a_i)^{-1}\delta_i^2 + O\Big(\sum_{i\neq j}\int_\Omega\nabla\delta_i\nabla\delta_j + 2\sum_{i\neq j}\int_\Omega\delta_i\delta_j\Big)$$

$$D = \int_\Omega K u^4 = \int_\Omega K(\sum_i K(a_i)^{-1/2}\delta_i)^4 = \sum_i K(a_i)^{-2}\int_\Omega K\delta_i^4 + O\Big(\sum_{i\neq j}\int_\Omega \delta_i^3\delta_j\Big)$$

On the other hand, we have

$$\int_\Omega |\nabla\delta_i|^2 = \frac{S_4}{2} - \frac{5}{3}\pi w_2\frac{\mathcal{H}(a_i)}{\lambda_i} + O(\frac{1}{\lambda_i^2}) \qquad \text{for } i\le q \tag{3.4}$$

$$\int_\Omega K\delta_i^4 = \frac{S_4}{2}K(a_i) - \frac{2}{3}\pi w_2\frac{\mathcal{H}(a_i)}{\lambda_i}K(a_i) - \frac{2c_3}{\lambda_i}\frac{\partial K}{\partial\nu}(a_i) + O\Big(\frac{1}{\lambda_i^2}\Big) \tag{3.5}$$

$$\int_\Omega |\nabla\delta_j|^2 = S_4 + O(\frac{1}{\lambda_j^2}) \qquad \text{for } j>q \tag{3.6}$$

$$\int_\Omega K\delta_j^4 = S_4 K(a_j) + O\Big(\frac{1}{\lambda_j^2}\Big) \tag{3.7}$$

$$\int_\Omega \nabla\delta_j\nabla\delta_i = O\Big(\varepsilon_{ij}\Big) \quad ; \quad \int_\Omega K\delta_j^3\delta_i = O\Big(\varepsilon_{ij}\Big) \quad ; \quad 3\int_\Omega K\delta_j\delta_i^3 = O\Big(\varepsilon_{ij}\Big) \tag{3.8}$$

We have also

$$\int_\Omega \delta_i^2 = O\Big(\frac{\log\lambda_i}{\lambda_i^2}\Big), \quad \text{and } \int_\Omega \delta_j\delta_i = O(\varepsilon_{ij}). \tag{3.9}$$

Thus,

$$\begin{aligned}
N =& \sum_{i\leq q} K(a_i)^{-1}\big(\frac{S_4}{2} - \frac{5}{3}\pi w_2 \frac{\mathcal{H}(a_i)}{\lambda_i}\big) + \sum_{i>q} K(a_i)^{-1} S_4 \\
&+ O\Big(\mu \sum_i \frac{\log\lambda_i}{\lambda_i^2} + \sum_{i\neq j}\varepsilon_{ij} + \sum_{i\leq q}\frac{1}{\lambda_i^2} + \sum_{i>q}\frac{1}{(\lambda_i d_i)^3}\Big) \\
=& \frac{S_4}{2}\big(\sum_{i\leq q} K(a_i)^{-1} + 2\sum_{i>q} K(a_i)^{-1}\big) - \frac{5}{3}\pi w_2 \sum_{i\leq q} K(a_i)^{-1}\frac{\mathcal{H}(a_i)}{\lambda_i} \\
&+ O\Big(\mu \sum_i \frac{\log\lambda_i}{\lambda_i^2} + \sum_{i\neq j}\varepsilon_{ij} + \sum_{i\leq q}\frac{1}{\lambda_i^2} + \sum_{i>q}\frac{1}{(\lambda_i d_i)^3}\Big) \\
=& \frac{S_4}{2}\Big(\sum_{i\leq q} K(a_i)^{-1} + 2\sum_{i>q} K(a_i)^{-1}\Big)\Big(1 - \frac{5\pi w_2}{3\theta}\sum_{i\leq q} K(a_i)^{-1}\frac{\mathcal{H}(a_i)}{\lambda_i}\Big) \\
&+ O\Big(\mu \sum_i \frac{\log\lambda_i}{\lambda_i^2} + \sum_{i\neq j}\varepsilon_{ij} + \sum_{i\leq q}\frac{1}{\lambda_i^2} + \sum_{i>q}\frac{1}{(\lambda_i d_i)^3}\Big)
\end{aligned}$$

and

$$\begin{aligned}
D =& \sum_{i\leq q} K(a_i)^{-2}\Big[K(a_i)\frac{S_4}{2} - \frac{2c_3}{\lambda_i}\frac{\partial K}{\partial\nu}(a_i) - \frac{2\pi w_2 K(a_i)}{3\lambda_i}\mathcal{H}(a_i)\Big] + \sum_{i>q} K(a_i)^{-1} S_4 + O\Big(\sum_{i\neq j}\int_\Omega \delta_i^3\delta_j\Big) \\
=& \frac{S_4}{2}\Big(\sum_{i\leq q} K(a_i)^{-1} + 2\sum_{i>q} K(a_i)^{-1}\Big) - \sum_{i\leq q} K(a_i)^{-2}\Big[\frac{2c_3}{\lambda_i}\frac{\partial K}{\partial\nu}(a_i) + \frac{2\pi w_2 K(a_i)}{3\lambda_i}\mathcal{H}(a_i)\Big] \\
&+ O\Big(\sum_{i\neq j}\int_\Omega \delta_i^3\delta_j\Big) \\
=& \frac{S_4}{2}\Big(\sum_{i\leq q} K(a_i)^{-1} + 2\sum_{i>q} K(a_i)^{-1}\Big)\Bigg(1 - \frac{2}{\theta}\sum_{i\leq q} K(a_i)^{-2}\Big[\frac{c_3}{\lambda_i}\frac{\partial K}{\partial\nu}(a_i) + \frac{\pi w_2 K(a_i)}{3\lambda_i}\mathcal{H}(a_i)\Big] \\
&+ O\Big(\sum_{i\neq j}\int_\Omega \delta_i^3\delta_j\Big)\Bigg)
\end{aligned}$$

So

$$\begin{aligned}
D^{-\frac{1}{2}} = \Big(\frac{S_4}{2}\Big[\sum_{i\leq q} K(a_i)^{-1} + 2\sum_{i>q} K(a_i)^{-1}\Big]\Big)^{-\frac{1}{2}}\Bigg(1 + \frac{1}{\theta}\sum_{i\leq q} K(a_i)^{-2}\Big[\frac{c_3}{\lambda_i}\frac{\partial K}{\partial\nu}(a_i) + \frac{\pi w_2 K(a_i)}{3\lambda_i}\mathcal{H}(a_i)\Big] \\
+ O\Big(\sum_{i\neq j}\int_\Omega \delta_i^3\delta_j\Big)\Bigg)
\end{aligned}$$

Using the formula of N and $D^{-1/2}$, the proof follows.

Proposition 3.2. *Let y_0 be defined in Theorem 1.1. For λ_0 large enough, we have*

$$J(\delta_{(y_0,\lambda_0)}) \leq c_\infty(1 - c\lambda_0^{-1})$$

Proof. From Proposition 3.1, we have

$$J(\delta_{(y_0,\lambda_0)}) = \Big(\frac{S_4}{2K(y_0)}\Big)^{1/2}\Big[1 + \frac{2}{S_4 K_1(y_0)\lambda_0}\Big(c_3\frac{\partial K}{\partial\nu}(y_0) - \frac{4\pi w_2}{3}K_1(y_0)\mathcal{H}(y_0)\Big) + O\Big(\mu\frac{\log\lambda_0}{\lambda_0^2}\Big)\Big]$$

where $\theta = \frac{S_4}{2K_1(y_0)}$.

Using the assumption (H_2), the proof follows.

Proposition 3.3. *For $a_i \in \Omega$ such that $\lambda_i d_i$ is very large, we have*

$$J(\delta_{(a_i,\lambda_i)}) = S_4^{1/2} K(a_i)^{-1/2}(1 + o(1)).$$

Proof. From Proposition 3.1, we have

$$J(\delta_{(a_i,\lambda_i)}) = \Big(\frac{S_4}{K(a_i)}\Big)^{1/2}\Big[1 + O\Big(\frac{1}{(\lambda_i d_i)^3} + \mu\frac{\log\lambda_i}{\lambda_i^2}\Big)\Big]$$

Hence, the proof follows.

4 Proof of Our Results

Proof of Theorem 1.1

Using the fact that $K_1(y_0) = \max K_1(y)$, we get

$$c_\infty = (S_4/2)K_1(y_0)^{-1/2} < (S_4/2)K_1(y)^{-1/2} \quad \text{for each} \quad y \in \partial\Omega.$$

In addition, the assumption (H_1) gives $c_\infty < S_4 K(y)^{-1/2}$ for each $y \in \Omega$.

Thus, we derive that all the levels of the critical points at infinity are above c_∞.

Furthermore, using Propositions 3.1, 3.2 and 3.3, we deduce that $J(u) \geq c_\infty(1 - c\varepsilon)$, $\forall\, u \in V(p,\varepsilon)$, $p \geq 1$. Hence, we can always choose ε such that for a fixed λ_0

$$J(\delta_{(y_0,\lambda_0)}) < J(u), \quad \forall\, u \in V(p,\varepsilon),\, p \geq 1. \tag{4.1}$$

We argue by contradiction, assuming that under the level $c_\infty(y_0)$ there is no solution of (P_μ).
Let $u(s)$ be the solution of the following equation

$$\frac{\partial u}{\partial s} = -\nabla J(u)\,, \quad u(0) = \frac{\delta_{(y_0,\lambda_0)}}{\| \delta_{(y_0,\lambda_0)} \|}$$

Observe that (4.1) implies that $u(s) \notin V(p,\varepsilon)$, for each $p \geq 1$.

Thus, for each $s \geq 0$, we have $| \nabla J(u(s)) | \geq c$ (c depends only on ε). Indeed, if there exists a subsequence (s_k) such that $\nabla J(u(s_k)) \to 0$ with the fact that $J(u(s_k))$ is bounded this implies that $u(s_k) \in V(p,\varepsilon)$. Therefore,

$$\frac{\partial J(u(s))}{\partial s} = -\mid \nabla J(u(s)) \mid^2 \leq -c^2\,, \quad \text{for each } s \geq 0.$$

Then, we get $J(u(s))$ goes to $-\infty$ when s goes to ∞, this we derive a contradiction.

Proof of Theorem 1.2

Using the fact that $K(y_1) = \max K(y)$, we get $d_\infty = S_4 K(y_1)^{-1/2} < S_4) K(y)^{-1/2}$ for each $y \in \Omega$. In addition, the assumption (H_1) gives $d_\infty < (S_4/2)K_1(y)^{-1/2}$ for each $y \in \partial\Omega$.

Thus, from Propositions 3.1, 3.2 and 3.3, we derive that, for a fixed λ_1, we can choose ε so that

$$J(u) > J(\delta_{(y_1,\lambda_1)}), \quad \text{for each } u \in V(p,\varepsilon),\, p \geq 1. \tag{4.2}$$

Now, we argue by contradiction and using the same argument in the proof of Theorem 1.1, it is easy to deduce the proof of Theorem 1.2.

5 Conclusion

Thus its been concluded that under some assumptions on the function K, there exists solutions of the nonlinear Neumann elliptic problem (P_μ) under levels defined at some boundary or interior points.

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Competing Interests

Author has declared that no competing interests exist.

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