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Mathematical Analysis of Plant Disease Dispersion Model that Incorporates wind Strength and Insect Vector at Equilibrium

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Authors' contributions

This work was carried out in collaboration between all authors. Author ALMM designed the study, performed the stability analysis, wrote the protocol and wrote the first draft of the manuscript. Author TO managed the analyses of the study. Author BO managed the literature searches. All authors read and approved the final manuscript.

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Abstract

Numerous plant diseases caused by pathogens like bacteria, viruses, fungi protozoa and pathogenic nematodes are propagated through media such as water, wind and other intermediary carries called vectors, and are therefore referred to as vector borne plant diseases.
Insect vector borne plant diseases are currently a major concern due to abundance of insects in the tropics which impacts negatively on food security, human health and world economies. Elimination or control of which can be achieved through understanding the process of propagation via Mathematical modeling. However existing models are linear and rarely incorporates climate change parameters to improve on their accuracy. Yields of plants can reduce significantly if they are infected by vectors borne diseases whose vectors have very short life span without necessarily inducing death to plants. Despite this, there is no reliable developed mathematical model to describe such dynamics.
This paper formulates and analyzes a dynamical nonlinear plant vector borne dispersion disease model

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that incorporates insect and plant population at equilibrium and wind as a parameter of climate change, to determine R_0, local and global stability in addition to sensitivity analysis of the basic reproduction number R_0.

Keywords: Basic reproduction number; sensitivity analysis; disease free equilibrium point (DFEP); endemic equilibrium point (EEP); local and global stability.

1 Introduction

Numerous plant diseases caused by pathogens namely: bacteria, viruses, fungi, protozoa and pathogenic nematodes are spread through media such as water, wind and other intermediary carriers called vectors, and are thus referred to as vector borne diseases [1].

Diseases that spread from one host to another are called infectious diseases, most of which are very destructive to plants and animals. Vector borne plant diseases have important transmission properties which include: survival rate of the vector, reproduction rate of the vector, time of the year and vectors / insects activity level specifically biting rate, rate of development and reproduction of pathogen on the vector amongst others [2].

The most common vectors are insects such as aphids, whiteflies, plant hoppers and leafhoppers with aphids as the most important group. Aphids have been estimated to cause 70% of the vector borne plant diseases [3].

Vectors (insects) are susceptible to external environmental influences which include the climate change and affect the spread of plant diseases. The climate parameters that vary include temperature, precipitation (humidity) and wind.

Climate change could first affect disease directly by either decreasing or increasing the encounter rate between the vectors and hosts by changing the ranges of the two species.

Disease severity should be positively correlated with increase in virulence and aggressiveness of the vector. These two effects on disease will be mediated by host resistance and encounter rates which are potentially affected by climate change.

Mathematical models provide a powerful tool used to understand the dynamics of disease spread through a population and in decision making in regards to disease prediction and disease control [4]. Empirical models like regression models and deterministic models with climate variables as predictors and epidemic parameters as response variables are used to predict the success of the hosts and vectors across the range of conditions with a possibility of extrapolation when the mechanism of the relationship is clearly understood. Simulation models based on theoretical relationships are used to predict outcomes under a range of scenario.

Kermack and McKendrick in [5] developed a classical model for micro parasite host interactions in mammals which forms the basis of plant epidemiological models. The first models of temporal development of epidemic plant diseases were developed by Van der Plank in [6] and have formed the basis for plant disease modeling. Subsequently, various models have been developed. Earlier theories on vector- borne diseases in [7,8,9,10,11] highlight the importance of the direct interactions between the vector's dynamics and disease prevalence in the host. The focus in this work is on the dynamics of the vector host disease system without considering the dynamics of the enemies of the vector.

There have also been many studies modeling specific crop diseases and biological controls. For example Tomato leave curl virus in India in [11] and Cassava mosaic virus in sub – Saharan Africa in [12] and in both cases the climate change factors were not considered.

Other researchers have studied the biological control of the vectors to reduce disease incidences using models. Gourley in [9] developed a model to investigate how pulsed application of biological larvicides or chemical insecticides on different life stages affects disease prevalence. Wei [13] modeled vector borne diseases with horizontal transmission and time delay while Cui developed and analyzed a simple vector- host epidemic model with direct transmission [14].

The above three models were extended by Lashari and Zama in their study of the global and back bifurcation of a vector borne disease model with horizontal transmission in host population [15]. Moore in [16] developed a model to determine how a predator of the vector affects the prevalence of a vector-borne disease in the absence of predators. Zhou in [4] improved on Moore's model [16] by developing a model to study the disease control threshold and limit cycles with persistence of disease or without disease. However, both [4,16] focus was on total host population dividing host and vector into susceptible and infectious. Lawrence and Wallace in [17] modeled the spatiotemporal dynamics of African Cassava Mosaic disease in which he incorporated wind but as an agent to the movement of the vector. Clearly the climate change variables have been modeled in isolation in any given model. We want to develop a general model to investigate the effect of a combination of the climate change factors on the spread of vector –borne plant diseases.

The research developed a Mathematical model incorporating the climatic change variables to be used to evaluate the dynamics of vector- borne diseases in plants. The climate change variables, temperature and precipitation influence the biting rate (a) are considered under the parameter for biting rate while wind was factored in as an agent to the movement of the vector, emigration or immigration hence incorporated in the model as $(\pm\theta)$. Climate induced death to insects was also examined. Since insects have very short life spans, the study developed a deterministic general model for the perennial plant disease with assumption that vectors reproduce very fast and attain equilibrium. The plant population is constant. Section 2 describes model development. Section 3 describes model analysis where also basic reproduction will be determined using the next generation method and analytical global stability by the Lyapnov method. Finally Section 4 expounds results.

2 Model Development

The model development begins with model description, model assumption, flow chart and listing of model equations.

2.1 Model description

Let $N_V(t)$ and $N_P(t)$ be the total population of the vectors and plants respectively. The plant population is subdivided into three compartment classes: susceptible plant class $S_P(t)$, exposed plant class $E_P(t)$ and infected plant class $I_P(t)$. The vector population is subdivided into two compartment classes: susceptible vector class $S_V(t)$ and infected plant class $I_V(t)$.

The recruitment rate of plants and vectors are given by μ_P and π respectively. The rates at which $E_P(t)$ recover naturally to $S_P(t)$ is given by ω. Climate induced death occur at a rate δ in $S_V(t)$ and $I_V(t)$ respectively. The constant natural death rate in subclasses $S_P(t), E_P(t)$ and $I_P(t)$ is given by μ_P. The rate at which $E_P(t)$ progresses to $I_P(t)$ is τ. The constant natural death rate in subclasses $S_V(t)$ and $I_V(t)$ is given by μ_V. The immigration and emigration rate of vector is given by θ, this implies that θ can be either be positive or negative. The infection rates for plants and vectors are β_1 and β_2 respectively while the vector biting rate is given by a.

2.2 Model assumptions

The model was formulated under the following assumptions;

i. Homogeneous mixing of the plants and vectors.
ii. It is possible for exposed plants to recover from natural immunity.
iii. The plant population is constant while vector population is not constant.
iv. Rate of immigration and emigration is constant and can be estimated.
v. Vector population is assumed to increase very fast and then attain equilibrium.
vi. Effects of competitions from other vectors are assumed to be insignificant.
vii. The susceptible plants are infected when they come into contact with infected insect vectors.
viii. Susceptible insect vectors become infected only when they get in contact with an infected plant and exposed plants. There is no vertical infection through birth and from insect to insect directly.

Modification parameter k is such that $k \geq 1$, implying that the climate factors increases force of infection.

2.3 Flow chart diagram and the model equations

We considered the five state variables and parameters described in section 2.1 and model assumptions listed in section 2.2 to represent the model flow chart diagram in the Fig. 1 in the next page;

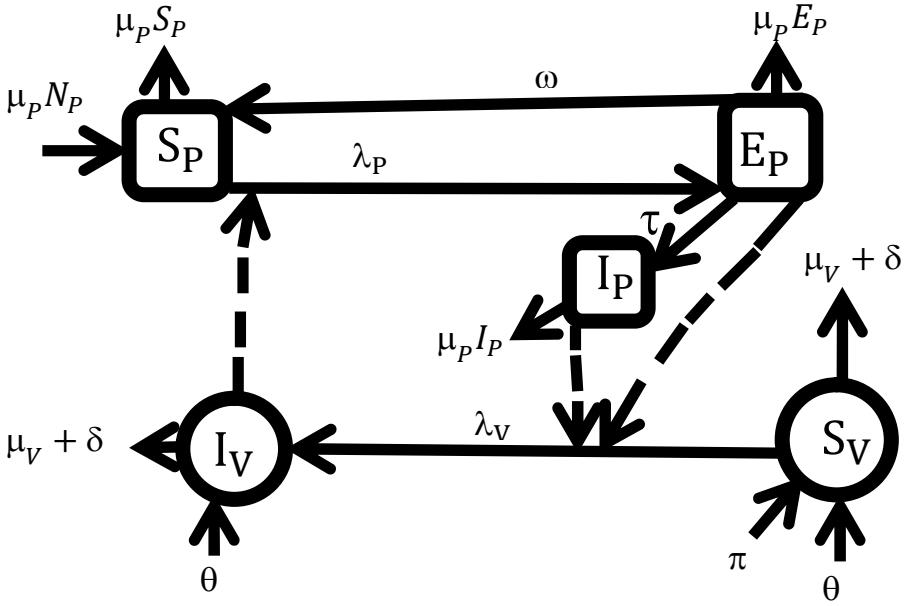

Fig. 1. Flow chart diagram of the $S\,E\,I\,S\,I$ model

The model was described by the following differential equations draw from the flow diagram

$$\frac{dS_P}{dt} = \mu_P N_P + \omega E_P - (k\lambda_P + \mu_P)S_P \quad (1)$$

$$\frac{dE_P}{dt} = k\lambda_P S_P - (\omega + \tau + \mu_P)E_P \quad (2)$$

$$\frac{dI_P}{dt} = \tau E_P - \mu_P I_P \tag{3}$$

$$\frac{dS_V}{dt} = \pi N_V \pm \theta S_V - (k\lambda_V + \delta + \mu_V)S_V \tag{4}$$

$$\frac{dI_V}{dt} = k\lambda_V S_V \pm \theta I_V - (\delta + \mu_V)I_V \tag{5}$$

where

$$N_V(t) = S_V(t) + I_V(t), N_P(t) = S_P(t) + E_P(t) + I_P(t), \lambda_P = a\beta_1 I_V \text{ and } \lambda_V = a\beta_2(E_P + \eta I_P).$$

The insects have very short life span, they reproduce within a very short time and therefore it is plausible to assume the dynamics of the insects analyzed at steady states. We equated equations $(4-5)$ to zero to obtain the expressions below at steady state,

$$N_V^* = S_V^* + I_V^*; S_V^* = \frac{\pi N_V^*}{k\lambda_V + \delta + \mu_V \pm \theta};\ I_V^* = \frac{k\lambda_V \pi N_V^*}{(k\lambda_V + \delta + \mu_V \pm \theta)(\delta + \mu_V \pm \theta)};$$
$$\lambda_P^* = \frac{a\beta_1 k\lambda_V \pi N_V^*}{(k\lambda_V + \delta + \mu_V \pm \theta)(\delta + \mu_V \pm \theta)}$$

After substituting λ_P^* at steady state, the system of equations$(1-5)$ reduces to;

$$\frac{dS_P}{dt} = \mu_P N_P + \omega E_P - \left(\frac{a\beta_1 K^2 \pi N_V^* \lambda_V}{(k\lambda_V + \delta + \mu_V \pm \theta)(\delta + \mu_V \pm \theta)} + \mu_P\right)S_P \tag{6}$$

$$\frac{dE_P}{dt} = \frac{a\beta_1 K^2 \pi N_V^* \lambda_V S_P}{(k\lambda_V + \delta + \mu_V \pm \theta)(\delta + \mu_V \pm \theta)} - (\omega + \tau + \mu_P)E_P \tag{7}$$

$$\frac{dI_P}{dt} = \tau E_P - \mu_P I_P \tag{8}$$

With initial conditions as, $S_P(0) = (S_P)_0$, $E_P(0) = (E_P)_0$ and $I_P(0) = (I_P)_0$. The sum of system of equations $[(6)-(8)]$ which is the rate of change of total population is given by $\frac{dN_P}{dt} = 0$.

3 Model Analysis

We analyzed the model by proving various theorems and carrying out algebraic computation dealing with different attributes.

3.1 Positivity and boundedness of the solutions

We proved positivity and boundedness by stating and proving the theorem below.

Theorem1: The region Z given by $Z = \{S_P(t), E_P(t), I_P(t) \in R_+^3; N_P = C\}$ is positively invariant and attracting with respect to model system $[(6)-(8)]$,

Proof.

Let $\{S_P(t), E_P(t) \text{ and } I_P(t)\}$ be any solutions of the system with non-negative initial conditions $\{S_P(0) \geq 0, E_P(0) \geq 0, I_P(0) \geq 0,\}$.

Since, $\frac{dS_P}{dt} = \mu_P N_P + \omega E_P - \left(\frac{a\beta_1 K^2 \pi N_V^* \lambda_V}{(k\lambda_V + \delta + \mu_V \pm \theta)(\delta + \mu_V \pm \theta)} + \mu_P\right) S_P$, it follows that $\frac{dS_P}{dt} = -\left(\frac{a\beta_1 K^2 \pi N_V^* \lambda_V}{(k\lambda_V + \delta + \mu_V \pm \theta)(\delta + \mu_V \pm \theta)} + \mu_P\right) S_P$. On integration, we obtained $S_P(t) = e^{\int_0^t -\left(\frac{a\beta_1 K^2 \pi N_V^* \lambda_V(s)}{(k\lambda_V(s) + \delta + \mu_V \pm \theta)(\delta + \mu_V \pm \theta)} + \mu_P\right) dt} \geq 0$. Clearly, $S_P(t) = e^{\int_0^t -\left(\frac{a\beta_1 K^2 \pi N_V^* \lambda_V(s)}{(k\lambda_V(s) + \delta + \mu_V \pm \theta)(\delta + \mu_V \pm \theta)} + \mu_P\right) dt}$ is a non-negative function of t, thus $S_P(t)$ stays positive.

The positivity of $E_P(t)$ and $I_P(t)$ was proved along the same lines hence considering equation (7) $\frac{dE_P}{dt} \geq -(\omega + \tau + \mu_P)E_P$ on integrating $E_P(t) = C_1 e^{-(\omega+\tau+\mu_P)t}$, where C_1 is a constant of integration. Applying initial condition we obtain $C_1 = E_P(0)$. Hence $E_P(t) = E_P(0)e^{-(\omega+\tau+\mu_P)t}$ and $E_P(t) = E_P(0)e^{-(\omega+\tau+\mu_P)t} \geq 0$.

Similarly, $I_P(t) = I_P(0)e^{-\mu_P t} \geq 0$.

Taking the time derivative of our total population along its solution path gives $\frac{dN_P}{dt} = \frac{dS_P}{dt} + \frac{dE_P}{dt} + \frac{dI_P}{dt} = 0$,

This implies that, $N_P(t) = C$, where C is the constant of integration.

Hence, $\lim_{t\to\infty} N(t) = C$

This proves the boundedness of the solutions inside Z and implies that all the solutions of our system $[(6) - (8)]$, starting in Z and will remains in Z for all $t \geq 0$. Thus Z is positively invariant and attracting, and hence it is sufficient to consider the dynamics of our system in Z. This completes the proof.

3.2 Disease-free equilibrium point (DFEP)

The disease-free equilibrium point (DFEP) of the system $[(6) - (8)]$, was obtained by setting all the E_P and I_P classes to zero to obtain $\mu N_P - \mu S_P^0 = 0$ which yields, $S_P^0 = N_P$.

The DFEP for our system is given by $E^0 = (S_P^0, E_P^0, I_P^0) = (N_P, 0, 0)$.

The DFEP (E^0) is the infection free equilibrium point of the system $[(6) - (8)]$, which indicates that in absence of insects, the system $[(6) - (8)]$ will consist of one compartment class.

3.3 The basic reproduction number (R_o)

We used the next-generation matrix method to determine the basic reproduction number (R_0) of the model Carlos Castillo-Chavez in [18]. Using the notation f for a matrix of new infections terms and v for the matrix of the remaining transfer of infection terms in our system, we obtained

$$f = \begin{pmatrix} \frac{a\beta_1 K^2 \pi N_V^* \lambda_V S_P}{(k\lambda_V + \delta + \mu_V \pm \theta)(\delta + \mu_V \pm \theta)} \\ 0 \end{pmatrix} \text{ and } v = \begin{pmatrix} (\omega + \tau + \mu_P)E_P \\ -\tau E_P + \mu_P I_P \end{pmatrix}.$$

We then obtained the matrices F and V by finding the Jacobian matrices of f and v evaluated at DFEP respectively.

$$F = \begin{pmatrix} \frac{a^2\beta_1\beta_2K^2\pi N_V^*\mathrm{N_P}}{(\delta+\mu_\mathrm{V}\pm\theta)^2} & \frac{a^2\beta_1\beta_2K^2\pi N_V^*\eta\mathrm{N_P}}{(\delta+\mu_\mathrm{V}\pm\theta)^2} \\ 0 & 0 \end{pmatrix} and\ V = \begin{pmatrix} \omega+\tau+\mu_\mathrm{P} & 0 \\ -\tau & \mu_\mathrm{P} \end{pmatrix}.$$

We computed V^{-1} the inverse of V hence $V^{-1} = \begin{pmatrix} \frac{1}{\omega+\tau+\mu_\mathrm{P}} & 0 \\ \frac{\tau}{\mu_\mathrm{P}(\omega+\tau+\mu_\mathrm{P})} & \frac{1}{\mu_\mathrm{P}} \end{pmatrix}$.

The eigenvalues of the matrix (FV^{-1}) are

$$\mathrm{q}(1) = 0\ \mathrm{and} \mathrm{q}(2) = \frac{a^2\beta_1\beta_2K^2\pi N_V^*\mathrm{N_P}}{(\delta+\mu_\mathrm{V}\pm\theta)^2(\omega+\tau+\mu_\mathrm{P})} + \frac{a^2\beta_1\beta_2K^2\pi N_V^*\eta\tau\mathrm{N_P}}{\mu_\mathrm{P}(\delta+\mu_\mathrm{V}\pm\theta)^2(\omega+\tau+\mu_\mathrm{P})}.$$

The basic reproduction number ($\mathrm{R_0}$) is given by the spectral radius ζ (the dominant eigenvalue) of the matrix FV^{-1} denoted by ζ hence

$$\mathrm{R_0} = \zeta(\mathrm{F}V^{-1}) = \frac{a^2\beta_1\beta_2K^2\pi N_V^*\mathrm{N_P}}{(\delta+\mu_\mathrm{V}\pm\theta)^2(\omega+\tau+\mu_\mathrm{P})}\left\{1+\frac{\eta\tau}{\mu_\mathrm{P}}\right\}.$$

The basic reproduction number($\mathrm{R_0}$) is the average number of susceptible plants which can be infected by an infected insect in absence of interventions.

3.4 Existence of endemic equilibrium point for the model (EEP)

We state and prove the following theorem

Theorem 2: A positive endemic equilibrium exists whenever $\mathrm{R_0} > 1$.

Proof.

Let the endemic equilibrium point be $E^* = (S_P^*, E_P^*, I_P^*)$ and the force of infection at equilibrium point be$\lambda_V^* = \mathrm{a}\beta_2(E_P^* + \eta I_P^*)$. The system of equation (6)– (8)were equated to zero then solved in terms of λ^* to obtain;

$$S_P^* = \frac{\mu_\mathrm{P}\mathrm{N_P}[\mathrm{k}\lambda_V^*(\delta+\mu_\mathrm{V}\pm\theta)+(\delta+\mu_\mathrm{V}\pm\theta)^2]}{\mathrm{a}\beta_1K^2\pi N_V^*\lambda_V^* + \mu_\mathrm{P}[\mathrm{k}\lambda_V^*(\delta+\mu_\mathrm{V}\pm\theta)+(\delta+\mu_\mathrm{V}\pm\theta)^2]} + \frac{[\mathrm{k}\lambda_V^*(\delta+\mu_\mathrm{V}\pm\theta)+(\delta+\mu_\mathrm{V}\pm\theta)^2]\omega E_P^*}{\mathrm{a}\beta_1K^2\pi N_V^*\lambda_V^* + \mu_\mathrm{P}[\mathrm{k}\lambda_V^*(\delta+\mu_\mathrm{V}\pm\theta)+(\delta+\mu_\mathrm{V}\pm\theta)^2]}$$

$$E_P^* = \frac{\mathrm{a}\beta_1K^2\pi N_V^*\lambda_V^*\mu_\mathrm{P}\mathrm{N_P}}{(\omega+\tau+\mu_\mathrm{P})\{\mathrm{a}\beta_1K^2\pi N_V^*\lambda_V^* + \mu_\mathrm{P}[\mathrm{k}\lambda_V^*(\delta+\mu_\mathrm{V}\pm\theta)+(\delta+\mu_\mathrm{V}\pm\theta)^2]\} - \mathrm{a}\beta_1K^2\pi N_V^*\lambda_\mathrm{V}\omega},$$

$$I_P^* = \frac{\tau\mathrm{a}\beta_1K^2\pi N_V^*\lambda_V^*\mu_\mathrm{P}\mathrm{N_P}}{\mu_\mathrm{P}[(\omega+\tau+\mu_\mathrm{P})\{\mathrm{a}\beta_1K^2\pi N_V^*\lambda_V^* + \mu_\mathrm{P}(\mathrm{k}\lambda_V^*(\delta+\mu_\mathrm{V}\pm\theta)+(\delta+\mu_\mathrm{V}\pm\theta)^2)\} - \mathrm{a}\beta_1K^2\pi N_V^*\lambda_\mathrm{V}\omega]},$$

$$\lambda_V^* = \frac{\lambda_V^* \mu_P R_0 (\delta + \mu_V \pm \theta)^2 (\omega + \tau + \mu_P)}{(\omega + \tau + \mu_P)\{a\beta_1 K^2 \pi N_V^* \lambda_V^* + \mu_P [k\lambda_V^* (\delta + \mu_V \pm \theta) + (\delta + \mu_V \pm \theta)^2]\} - a\beta_1 K^2 \pi N_V^* \lambda_V \omega}$$

$\lambda_V^* = 0$, Corresponds to DFE, while the expression for λ_V^* below correspond to endemic equilibrium point.

$$\lambda_V^* = \frac{\mu_P (\delta + \mu_V \pm \theta)^2 (\omega + \tau + \mu_P)(R_0 - 1)}{(\omega + \tau + \mu_P)\{a\beta_1 K^2 \pi N_V^* + \mu_P [k(\delta + \mu_V \pm \theta)]\} - a\beta_1 K^2 \pi N_V^* \omega}$$

Since $(\omega + \tau + \mu_P)\{a\beta_1 K^2 \pi N_V^* + \mu_P [k(\delta + \mu_V \pm \theta)]\} - a\beta_1 K^2 \pi N_V^* \omega > 0$, the condition necessary and sufficient for $\lambda_V^* > 0$ is $\mu_P (\delta + \mu_V \pm \theta)^2 (\omega + \tau + \mu_P)(R_0 - 1) > 0$ that is $R_0 > 1$. That completes the proof.

The endemic equilibrium point (EEP) for our system $E^* = (S_P^*, E_P^*, I_P^*)$ is obtained by substituting, $\lambda_V^* = \frac{\mu_P (\delta + \mu_V \pm \theta)^2 (\omega + \tau + \mu_P)(R_0 - 1)}{(\omega + \tau + \mu_P)\{a\beta_1 K^2 \pi N_V^* + \mu_P [k(\delta + \mu_V \pm \theta)]\} - a\beta_1 K^2 \pi N_V^* \omega}$ in S_P^*, E_P^* *and* I_P^*.

3.5 Local stability of the disease free equilibrium point (DFE)

To determine the local stability of the disease free equilibrium point we stated and proved the following theorem.

Theorem 3: The DFEP of the system $[(6) - (8)]$ is locally asymptotically stable $R_0 < 1$ and unstable otherwise.

Proof.

To establish the local stability of the system$[(6) - (8)]$, we used the Jacobian of the model evaluated at E_*^0. Stability of this steady state was then determined based on the signs of eigenvalues of the corresponding Jacobian which are functions of the model parameters. The Jacobian matrix evaluated at disease free equilibrium point E^0 is obtained as

$$J(E^0) = \begin{pmatrix} -\frac{a\beta_1 K^2 \pi N_V^* N_P}{(\delta + \mu_V \pm \theta)^2} + \mu_P & -\frac{a^2 \beta_1 \beta_2 K^2 \pi N_V^* N_P}{(\delta + \mu_V \pm \theta)^2} + \omega & -\frac{a^2 \beta_1 \beta_2 \eta K^2 \pi N_V^* N_P}{(\delta + \mu_V \pm \theta)^2} \\ 0 & \frac{a^2 \beta_1 \beta_2 K^2 \pi N_V^* N_P}{(\delta + \mu_V \pm \theta)^2} - (\omega + \tau + \mu_P) & \frac{a^2 \beta_1 \beta_2 \eta K^2 \pi N_V^* N_P}{(\delta + \mu_V \pm \theta)^2} \\ 0 & \tau & -\mu_P \end{pmatrix}$$

Using the Mathematica software the eigenvalues of the matrix $J(E^0)$ are;

$$q(1) = -\frac{a\beta_1 K^2 \pi N_V^* N_P}{(\delta + \mu_V \pm \theta)^2} + \mu_P,$$

$$q(2) = \frac{1}{2}(-\tau - \omega - 2\mu_P + \frac{R_0 \mu_P (\tau + \omega + \mu_P)}{\eta\tau + \mu_P} - \sqrt{(\tau + \omega)^2 + \frac{2((-1 + 2\eta)\tau - \omega) R_0 \mu_P (\tau + \omega + \mu_P)}{\eta\tau + \mu_P} + \frac{{R_0}^2 {\mu_P}^2 (\tau + \omega + \mu_P)^2}{(\eta\tau + \mu_P)^2}}),$$

$$q(3) = \frac{1}{2}(-\tau - \omega - 2\mu_P + \frac{R_0\mu_P(\tau+\omega+\mu_P)}{\eta\tau+\mu_P}) + \sqrt{(\tau+\omega)^2 + \frac{2((-1+2\eta)\tau-\omega)R_0\mu_P(\tau+\omega+\mu_P)}{\eta\tau+\mu_P} + \frac{{R_0}^2{\mu_P}^2(\tau+\omega+\mu_P)^2}{(\eta\tau+\mu_P)^2}}$$

Clearly, $q(1) < 0$. We used Mathematica software to make R_0 the subject of the expressions for $q(2)$ and $q(3)$ as below,

$$\frac{1}{2}(-\tau - \omega - 2\mu_P + \frac{R_0\mu_P(\tau+\omega+\mu_P)}{\eta\tau+\mu_P}) < \pm\sqrt{(\tau+\omega)^2 + \frac{2((-1+2\eta)\tau-\omega)R_0\mu_P(\tau+\omega+\mu_P)}{\eta\tau+\mu_P} + \frac{{R_0}^2{\mu_P}^2(\tau+\omega+\mu_P)^2}{(\eta\tau+\mu_P)^2}}$$

We found that condition necessary and sufficient for $q(2)\ and\ q(3)$ to be less than zero is $R_0 < 1$. This completed the proof.

3.6 Global stability of the disease free point

To prove the global stability, we stated and proved the following theorem.

Theorem 4: The conditions necessary and sufficient for the DFE to be globally asymptotically stable in Lyapunov senseare $\left\{\frac{\mu_P}{\tau}+\eta\right\} \le 0$ or $\frac{a\beta_1K^2\pi N_V^*S_P^0I_P}{(k(E_P+\eta I_P)+\delta+\mu_V\pm\theta)(\delta+\mu_V\pm\theta)}\left\{\frac{\mu_P}{\tau}+\eta\right\} < \mu_P\frac{(S_P-S_P^0)^2}{S_P} + \mu_P I_P + I_P\frac{{\mu_P}^2}{\tau} + \mu_P N_P\frac{S_P^0}{S_P} + \omega E_P\frac{S_P^0}{S_P}$ and unstable otherwise.

Proof.

We proposed the following Lyapunov function for the system $[(6)-(8)]$

$$L(S_P, E_P, I_P) = S_P - S_P^0 - S_P^0 ln\frac{S_P}{S_P^0} + E_P + X_1 I_P \quad (9)$$

$L(S_P, E_P, I_P)$ is positive definite and satisfies the conditions. For $\frac{dL(S_P,E_P,I_P)}{dt}$ to be negative definite, it must satisfy

$$\frac{dL(S_P^0, E_P^0, I_P^0)}{dt} = 0 \text{ and } \frac{dL(S_P, E_P, I_P)}{dt} < 0$$

Where, X_1 and X_2 were positive constants to be determined. The time derivative of the Lyapunov function is obtained as,

$$\frac{dL(S_P,E_P,I_P)}{dt} = \left(1-\frac{S_P^0}{S_P}\right)\frac{dS_P}{dt} + X_1\left(1-\frac{E_P^0}{E_P}\right)\frac{dE_P}{dt} + X_2\frac{dI_P}{dt}, \quad (10)$$

On substituting for $\frac{dS_P}{dt}, \frac{dE_P}{dt}$ and $\frac{dI_P}{dt}$ in equations (6) -(8) and considering $\mu_P N_P = \mu_P S_P^0$ we obtained;

$$\frac{dL(S_P, E_P, I_P)}{dt} = -\mu_P \frac{(S_P - S_P^0)^2}{S_P} - (\tau + \mu_P)E_P + X_1\{\tau E_P - \mu_P I_P\} - \mu_P N_P \frac{S_P^0}{S_P} - \omega E_P \frac{S_P^0}{S_P} + \left(\frac{a\beta_1 K^2 \pi N_V^* (E_P + \eta I_P)}{(k(E_P + \eta I_P) + \delta + \mu_V \pm \theta)(\delta + \mu_V \pm \theta)}\right) S_P^0$$

Setting E_P to zero we obtain

$$-(\tau + \mu_P) + \frac{a\beta_1 K^2 \pi N_V^* S_P^0}{(k(E_P + \eta I_P) + \delta + \mu_V \pm \theta)(\delta + \mu_V \pm \theta)} + X_1 \tau = 0.$$

$$X_1 = 1 + \frac{\mu_P}{\tau} - \frac{a\beta_1 K^2 \pi N_V^* S_P^0}{\tau(k(E_P + \eta I_P) + \delta + \mu_V \pm \theta)(\delta + \mu_V \pm \theta)}$$

Then

$$\frac{dL(S_P, E_P, I_P)}{dt} = -\mu_P \frac{(S_P - S_P^0)^2}{S_P} - \mu_P I_P - I_P \frac{{\mu_P}^2}{\tau} - \mu_P N_P \frac{S_P^0}{S_P} - \omega E_P \frac{S_P^0}{S_P} + \frac{a\beta_1 K^2 \pi N_V^* S_P^0 I_P}{(k(E_P + \eta I_P) + \delta + \mu_V \pm \theta)(\delta + \mu_V \pm \theta)}\left\{\frac{\mu_P}{\tau} + \eta\right\},$$

The conditions necessary and sufficient for $\frac{dL(S_P, E_P, I_P)}{dt} < 0$ are
$\left\{\frac{\mu_P}{\tau} + \eta\right\} \leq 0$ or $+\frac{a\beta_1 K^2 \pi N_V^* S_P^0 I_P}{(k(E_P+\eta I_P)+\delta+\mu_V \pm \theta)(\delta+\mu_V \pm \theta)}\left\{\frac{\mu_P}{\tau} + \eta\right\} < \mu_P \frac{(S_P - S_P^0)^2}{S_P} + \mu_P I_P + I_P \frac{{\mu_P}^2}{\tau} + \mu_P N_P \frac{S_P^0}{S_P} + \omega E_P \frac{S_P^0}{S_P}$.
This completed the proof.

3.7 Bifurcation analysis

This bifurcation was explored using the Centre Manifold theory in [19]. The change of variables was first made for simplicity. Let $S_P = y_1, E_P = y_2$ and $I_P = y_4$, so that,

$$N_P = y_1 + y_2 + y_3.$$

Further, by using vector notation, $y = (y_1, y_2, y_3)^T$, the insect vector disease model$[(6) - (8)]$was written in the form $\frac{dy}{dt} = F(y)$, with $F = (p_1, p_2, p_3)^T$, as follows:

$$\dot{y_1} = p_1 = \mu_P N_P + \omega y_2 - \left(\frac{a\beta_1 K^2 \pi N_V^* \lambda_V^{**}}{(k\lambda_V^{**} + \delta + \mu_V \pm \theta)(\delta + \mu_V \pm \theta)} + \mu_P\right) y_1, \quad (11)$$

$$\dot{y_2} = p_2 = \frac{a\beta_1 K^2 \pi N_V^* \lambda_V^{**} y_1}{(k\lambda_V^{**} + \delta + \mu_V \pm \theta)(\delta + \mu_V \pm \theta)} - (\omega + \tau + \mu_P) y_2 \quad (12)$$

$$\dot{y_3} = p_3 = \tau y_2 - \mu_P y_3 \quad (13)$$

With $\lambda_V^{**} = a\beta_2(y_2 + \eta y_3)$.

The method entailed evaluating the Jacobian of the system $[(9)-(11)]$ at the disease free equilibrium point, by$J(E_*^0)$, has eigenvalues$\left(-\frac{(\omega+\tau+\mu_P)\mu_P}{a^2\beta_2(\mu_P+\eta\tau)}+\mu_P,-\frac{(\omega+\tau+\mu_P)\eta\tau+\eta\tau\mu_P+{\mu_P}^2}{\mu_P+\eta\tau}\ ,\ 0\right)$, denoted by $J(E_*^0)$. This gave:

$$J(E_*^0).=\begin{pmatrix}-\frac{a\beta_1^*K^2\pi N_V^*N_P}{(\delta+\mu_V\pm\theta)^2}+\mu_P & -\frac{a^2\beta_1^*\beta_2K^2\pi N_V^*N_P}{(\delta+\mu_V\pm\theta)^2}+\omega & -\frac{a^2\beta_1^*\beta_2\eta K^2\pi N_V^*N_P}{(\delta+\mu_V\pm\theta)^2}\\ 0 & \frac{a^2\beta_1^*\beta_2K^2\pi N_V^*N_P}{(\delta+\mu_V\pm\theta)^2}-(\omega+\tau+\mu_P) & \frac{a^2\beta_1^*\beta_2\eta K^2\pi N_V^*N_P}{(\delta+\mu_V\pm\theta)^2}\\ 0 & \tau & -\mu_P\end{pmatrix}$$

We considered the case where $R_0=1$. Suppose, further, that $\beta_1=\beta_1^*$is chosen as a bifurcation parameter. Solving for β^* from $R_0^*=1$ gives

$$\beta_1^*=\frac{(\delta+\mu_V\pm\theta)^2(\omega+\tau+\mu_P)\mu_P}{a^2\beta_2K^2\pi N_V^*N_P(\mu_P+\eta\tau)}$$

Using Mathematica software the Jacobian of $\frac{dy}{dt}=F(y)$ at the disease free equilibrium point, with $\beta=\beta^*$, denoted by $J(E_*^0)$, has eigenvalues $\left(-\frac{(\omega+\tau+\mu_P)\mu_P}{a^2\beta_2(\mu_P+\eta\tau)}+\mu_P,-\frac{(\omega+\tau+\mu_P)\eta\tau+\eta\tau\mu_P+{\mu_P}^2}{\mu_P+\eta\tau},\ 0\right)$. We obtained one zero eigenvalue and two negative eigenvalues hence, the Centre Manifold theory was used to analyze the dynamics of the model [19].

Eigenvectors of J_{β^*}: For the case, when $R_0=1$ it can be shown that the Jacobian$[J(E_*^0)]$ at $\beta_1=\beta_1^*$(denoted byJ_{β^*}) has a right eigenvector given by $u=[u_1,u_2,u_3]^T$, where,

$$u_1=\frac{-\frac{a^2\beta_1^*\beta_2K^2\pi N_V^*N_Pu_2}{(\delta+\mu_V\pm\theta)^2}-\frac{a^2\beta_1^*\beta_2\eta K^2\pi N_V^*N_Pu_3}{(\delta+\mu_V\pm\theta)^2}+\omega u_2}{\frac{a\beta_1^*K^2\pi N_V^*N_P}{(\delta+\mu_V\pm\theta)^2}-\mu_P}<0,u_2=u_2>0,u_3=\frac{\tau u_2}{\mu_P}>0,$$

Further, J_{β^*} has a left eigenvector($v=[v_1,v_2,v_3]$), where,

$$v_1=0,\ v_2=v_2>0,\qquad v_3=\frac{\frac{a^2\beta_1^*\beta_2\eta K^2\pi N_V^*N_Pv_2}{(\delta+\mu_V\pm\theta)^2}}{\mu_P}>0.$$

Since($v_1=0$), we only need to compute the partial derivatives of p_2 and p_3 (at the disease free equilibrium point). For the system $[(9)-(11)]$the associated non-zero partial derivative of f_3 (at the disease free equilibrium) is given by

$$\frac{\partial^2p_2}{\partial y_1\,\partial y_2}=\frac{\partial^2p_2}{\partial p_2\,\partial y_1}=\frac{a^2\beta_1\beta_2K^2\pi N_V^*}{(\delta+\mu_V\pm\theta)^2}\ \text{and}\ \frac{\partial^2p_3}{\partial y_1\,\partial y_3}=\frac{\partial^2p_3}{\partial y_3\,\partial y_1}=\frac{a^2\beta_1\beta_2\eta K^2\pi N_V^*}{(\delta+\mu_V\pm\theta)^2}.$$

This implies that,

$$r=\sum_{i,j=1}^{3}v_2u_iu_j\frac{\partial^2p_2}{\partial x_i\,\partial x_j},$$

$$\mathrm{r}=\left\{\mathrm{v}_2\mathrm{u}_1\mathrm{u}_2\frac{\mathrm{a}^2\beta_1\beta_2K^2\pi N_V^*}{\left(\delta+\mu_\mathrm{V}\pm\theta\right)^2}+\mathrm{v}_2\mathrm{u}_2\mathrm{u}_1\frac{\mathrm{a}^2\beta_1\beta_2K^2\pi N_V^*}{\left(\delta+\mu_\mathrm{V}\pm\theta\right)^2}+\mathrm{v}_2\mathrm{u}_1\mathrm{u}_3\frac{\mathrm{a}^2\beta_1\beta_2\eta K^2\pi N_V^*}{\left(\delta+\mu_\mathrm{V}\pm\theta\right)^2}+\mathrm{v}_2\mathrm{u}_3\mathrm{u}_1\frac{\mathrm{a}^2\beta_1\beta_2\eta K^2\pi N_V^*}{\left(\delta+\mu_\mathrm{V}\pm\theta\right)^2}\right\}.$$

Since u_1 are less than zero, it follows that,

$$\mathrm{r}=2\mathrm{v}_2\left\{\mathrm{u}_1\mathrm{u}_2\frac{\partial^2\mathrm{p}_2}{\partial\mathrm{y}_1\,\partial\mathrm{y}_2}+\mathrm{u}_1\mathrm{u}_3\frac{\partial^2\mathrm{p}_2}{\partial\mathrm{y}_1\,\partial\mathrm{y}_3}\right\}<0.$$

Also,

$$q=\sum_{k,ij=1}^{n}v_ku_i\frac{\partial^2p_k}{\partial y_i\partial\beta^*}(0,0)$$

$$\frac{\partial^2\mathrm{p}_2}{\partial\mathrm{y}_2\,\partial\beta_1^*}=\frac{\mathrm{a}^2\beta_2K^2\pi N_V^*\mathrm{N_P}}{(\delta+\mu_\mathrm{V}\pm\theta)^2};\ \frac{\partial^2\mathrm{p}_2}{\partial\mathrm{y}_3\,\partial\beta_1^*}=\frac{\mathrm{a}^2\beta_2K^2\pi\eta N_V^*\mathrm{N_P}}{(\delta+\mu_\mathrm{V}\pm\theta)^2}$$

$$\mathrm{q}=\sum_{\mathrm{i}=1}^{3}\mathrm{v}_2\mathrm{u}_\mathrm{i}\frac{\partial^2\mathrm{p}_2}{\partial\mathrm{y}_\mathrm{i}\,\partial\beta_1^*}=\mathrm{v}_2\mathrm{u}_2\frac{\partial^2\mathrm{p}_2}{\partial\mathrm{y}_2\,\partial\beta_1^*}+\mathrm{v}_2\mathrm{u}_3\frac{\partial^2\mathrm{p}_2}{\partial\mathrm{y}_3\,\partial\beta_1^*},$$

$$\mathrm{q}=2\mathrm{v}_2\left\{\mathrm{u}_2\frac{\mathrm{a}^2\beta_2K^2\pi N_V^*\mathrm{N_P}}{(\delta+\mu_\mathrm{V}\pm\theta)^2}+\mathrm{u}_3\frac{\mathrm{a}^2\beta_2K^2\pi\eta N_V^*\mathrm{N_P}}{(\delta+\mu_\mathrm{V}\pm\theta)^2}\right\}.$$

Since $\mathrm{v}_2,\mathrm{u}_2$ and u_3 were greater than zero it followed that, $\mathrm{q}>0$.

Hence, from the theorem of Castillo-Chavez and Song in [18] and considering the following general system of ordinary differential equations with a parameter β_1^* it follows that when $\beta_1^*<0$ with $\left|\beta_1^*\right|\ll1,(0,0)$ is unstable, and there exists a negative and locally asymptotically stable equilibrium; when $0<\beta_1^*\ll1,(0,0)$ is stable and there exists a positive unstable equilibrium.

3.8 Global stability of the endemic equilibrium point (EEP)

To determine global stability we stated and proved the following theorem.

Theorem 6: The conditions necessary and sufficient for the EEP to be globally asymptotically stable in Lyapunov sense is $\frac{\mathrm{dK(S_P,E_P,I_P)}}{\mathrm{dt}}\leq0$ is $P>Q$ and unstable otherwise,

where,

$$\mathrm{P}=\mu_\mathrm{P}\frac{(\mathrm{S_P}-S_P^*)^2}{\mathrm{S_P}}+\{1+\frac{\mu_\mathrm{P}}{\tau}\}\mu_\mathrm{P}\mathrm{I_P}\}+\frac{\mathrm{a}\beta_1K^2\pi N_V^*(\mathrm{E_P}+\eta\mathrm{I_P})\mathrm{S_P}}{\left(\mathrm{k}\lambda_\mathrm{V}+\delta+\mu_\mathrm{V}\pm\theta\right)\left(\delta+\mu_\mathrm{V}\pm\theta\right)}+\left(\frac{a^2\beta_1\beta_2K^2\pi N_V^*(E_P^*+\eta I_P^*)}{\left(\mathrm{ka}\beta_2(E_P^*+\eta I_P^*)+\delta+\mu_\mathrm{V}\pm\theta\right)\left(\delta+\mu_\mathrm{V}\pm\theta\right)}\right)S_P^*\frac{S_P^*}{\mathrm{S_P}}+\omega\frac{S_P^*}{\mathrm{S_P}}\mathrm{E_P}+\{1+\frac{\mu_\mathrm{P}}{\tau}\}\tau\frac{I_P^*\mathrm{E_P}}{\mathrm{I_P}},$$

$$Q=\frac{a^2\beta_1\beta_2K^2\pi N_V^*S_P^*}{\left(\mathrm{ka}\beta_2(\mathrm{E_P}+\eta\mathrm{I_P})+\delta+\mu_\mathrm{V}\pm\theta\right)\left(\delta+\mu_\mathrm{V}\pm\theta\right)}\{\mathrm{E_P}+\eta\mathrm{I_P}\frac{\mathrm{S_P}E_P^*}{\mathrm{E_P}}+\omega\frac{S_P^*E_P^*}{\mathrm{S_P}}+\{1+\frac{\mu_\mathrm{P}}{\tau}\}\mu_\mathrm{P}I_P^*+(\tau+\mu_\mathrm{P})E_P^*+\frac{a^2\beta_1\beta_2K^2\pi N_V^*(E_P^*+\eta I_P^*)S_P^*}{\left(\mathrm{ka}\beta_2(E_P^*+\eta I_P^*)+\delta+\mu_\mathrm{V}\pm\theta\right)\left(\delta+\mu_\mathrm{V}\pm\theta\right)}$$

Proof.

We proposed the following Lyapunov function,

$$\mathrm{K}(\mathrm{S_P},\mathrm{E_P},\mathrm{I_P}) = \mathrm{S_P} - S_P^* - S_P^* Ln\frac{\mathrm{S_P}}{S_P^*} + \left(\mathrm{E_P} - E_P^* - E_P^* Ln\frac{\mathrm{E_P}}{E_P^*}\right) + \mathrm{X_1}\left(\mathrm{I_P} - I_P^* - I_P^* Ln\frac{\mathrm{I_P}}{I_P^*}\right),$$
$$E^* = (S_P^*, E_P^*, I_P^*)$$

where, $\mathrm{X_1}$ is a positive constant to be determined. The Lyapunov function $\mathrm{K}(\mathrm{S_P},\mathrm{E_P},\mathrm{I_P})$ satisfies the conditions, $K(S_P^*, E_P^*, I_P^*) = 0\ and\ \mathrm{K}(\mathrm{S_P},\mathrm{E_P},\mathrm{I_P}) > 0$, hence it is positive definite. For $\frac{\mathrm{dK}(\mathrm{S_P},\mathrm{E_P},\mathrm{I_P})}{\mathrm{dt}}$ to be negative definite, it must satisfy,

$$\frac{\mathrm{dK}(S_P^*, E_P^*, I_P^*)}{\mathrm{dt}} = 0 \quad \text{and} \quad \frac{\mathrm{dK}(\mathrm{S_P},\mathrm{E_P},\mathrm{I_P})}{\mathrm{dt}} < 0.$$

Determining the time derivative of the lyapunov equation we obtained,

$$\frac{\mathrm{dK}(\mathrm{S_P},\mathrm{E_P},\mathrm{I_P})}{\mathrm{dt}} = \left(1 - \frac{S_P^*}{\mathrm{S_P}}\right)\frac{\mathrm{dS_P}}{\mathrm{dt}} + \left(1 - \frac{E_P^*}{\mathrm{E_P}}\right)\frac{\mathrm{dE_P}}{\mathrm{dt}} + X_1\left(1 - \frac{I_P^*}{\mathrm{I_P}}\right)\frac{\mathrm{dI_P}}{\mathrm{dt}},$$

Substituting for $\frac{\mathrm{dS_P}}{\mathrm{dt}}, \frac{\mathrm{dE_P}}{\mathrm{dt}}$ and $\frac{\mathrm{dI_P}}{\mathrm{dt}}$ and considering that at endemic equilibrium point $E^* = (S_P^*, E_P^*, I_P^*)$ for the system satisfies equations (6) - (8)

$$\mu_\mathrm{P}\mathrm{N_P} = -\omega E_P^* + \left(\frac{\mathrm{a}\beta_1 K^2 \pi N_V^* \lambda_V^*}{(\mathrm{k}\lambda_V^* + \delta + \mu_\mathrm{V} \pm \theta)(\delta + \mu_\mathrm{V} \pm \theta)} + \mu_\mathrm{P}\right) S_P^*,$$

$$\frac{\mathrm{a}\beta_1 K^2 \pi N_V^* \lambda_V^* S_P^*}{(\mathrm{k}\lambda_V^* + \delta + \mu_\mathrm{V} \pm \theta)(\delta + \mu_\mathrm{V} \pm \theta)} = (\omega + \tau + \mu_\mathrm{P})E_P^*,$$

$\tau E_P^* = \mu_\mathrm{P} I_P^*$ and $\lambda_V^* = \mathrm{a}\beta_2(E_P^* + \eta I_P^*)$, we obtained,

$$\begin{aligned}
\frac{\mathrm{dK}(\mathrm{S_P},\mathrm{E_P},\mathrm{I_P})}{\mathrm{dt}} &= -\mu_\mathrm{P}\frac{(\mathrm{S_P} - S_P^*)^2}{\mathrm{S_P}} + \{-(\tau + \mu_\mathrm{P}) + X_1\tau\}\mathrm{E_P} \\
&+ \frac{a^2\beta_1\beta_2 K^2 \pi N_V^*(E_P^* + \eta I_P^*)S_P^*}{(\mathrm{ka}\beta_2(E_P^* + \eta I_P^*) + \delta + \mu_\mathrm{V} \pm \theta)(\delta + \mu_\mathrm{V} \pm \theta)} - X_1\mu_\mathrm{P}\mathrm{I_P} \\
&+ \frac{a^2\beta_1\beta_2 K^2 \pi N_V^* S_P^*}{(\mathrm{ka}\beta_2(\mathrm{E_P} + \eta\mathrm{I_P}) + \delta + \mu_\mathrm{V} \pm \theta)(\delta + \mu_\mathrm{V} \pm \theta)}\{\mathrm{E_P} + \eta\mathrm{I_P}\} \\
&- \frac{\mathrm{a}\beta_1 K^2 \pi N_V^*(\mathrm{E_P} + \eta\mathrm{I_P})\mathrm{S_P}}{(\mathrm{k}\lambda_\mathrm{V} + \delta + \mu_\mathrm{V} \pm \theta)(\delta + \mu_\mathrm{V} \pm \theta)}\frac{\mathrm{S_P}E_P^*}{\mathrm{E_P}} \\
&- \left(\frac{a^2\beta_1\beta_2 K^2 \pi N_V^*(E_P^* + \eta I_P^*)}{(\mathrm{ka}\beta_2(E_P^* + \eta I_P^*) + \delta + \mu_\mathrm{V} \pm \theta)(\delta + \mu_\mathrm{V} \pm \theta)}\right) S_P^*\frac{S_P^*}{\mathrm{S_P}} - \omega\frac{S_P^*}{\mathrm{S_P}}\mathrm{E_P} - X_1\tau\frac{I_P^*\mathrm{E_P}}{\mathrm{I_P}} \\
&+ \omega\frac{S_P^* E_P^*}{\mathrm{S_P}} + X_1\mu_\mathrm{P} I_P^* + (\tau + \mu_\mathrm{P})E_P^*,
\end{aligned}$$

Setting $\mathrm{E_P}$ to zero we obtained the following equation,

$$-(\tau + \mu_\mathrm{P}) + X_1\tau = 0;\ X_1 = \{1 + \frac{\mu_\mathrm{P}}{\tau}\}.$$

Hence,

$$\frac{\mathrm{dK}(\mathrm{S_P},\mathrm{E_P},\mathrm{I_P})}{\mathrm{dt}}$$
$$= -\mu_{\mathrm{P}}\frac{(\mathrm{S_P}-S_P^*)^2}{\mathrm{S_P}} - \{1+\frac{\mu_{\mathrm{P}}}{\tau}\}\mu_{\mathrm{P}}\mathrm{I_P}\} - \frac{\mathrm{a}\beta_1 K^2\pi N_V^*(\mathrm{E_P}+\eta\mathrm{I_P})\mathrm{S_P}}{(\mathrm{k}\lambda_{\mathrm{V}}+\delta+\mu_{\mathrm{V}}\pm\theta)(\delta+\mu_{\mathrm{V}}\pm\theta)}$$
$$-\left(\frac{a^2\beta_1\beta_2 K^2\pi N_V^*(E_P^*+\eta I_P^*)}{(\mathrm{ka}\beta_2(E_P^*+\eta I_P^*)+\delta+\mu_{\mathrm{V}}\pm\theta)(\delta+\mu_{\mathrm{V}}\pm\theta)}\right)S_P^*\frac{S_P^*}{\mathrm{S_P}} - \omega\frac{S_P^*}{\mathrm{S_P}}\mathrm{E_P} - \{1+\frac{\mu_{\mathrm{P}}}{\tau}\}\tau\frac{I_P^*\mathrm{E_P}}{\mathrm{I_P}}$$
$$+\frac{a^2\beta_1\beta_2 K^2\pi N_V^* S_P^*}{(\mathrm{ka}\beta_2(\mathrm{E_P}+\eta\mathrm{I_P})+\delta+\mu_{\mathrm{V}}\pm\theta)(\delta+\mu_{\mathrm{V}}\pm\theta)}\{\mathrm{E_P}+\eta\mathrm{I_P}\frac{\mathrm{S_P}E_P^*}{\mathrm{E_P}}+\omega\frac{S_P^*E_P^*}{\mathrm{S_P}}+\{1$$
$$+\frac{\mu_{\mathrm{P}}}{\tau}\}\mu_{\mathrm{P}}I_P^*+(\tau+\mu_{\mathrm{P}})E_P^*+\frac{a^2\beta_1\beta_2 K^2\pi N_V^*(E_P^*+\eta I_P^*)S_P^*}{(\mathrm{ka}\beta_2(E_P^*+\eta I_P^*)+\delta+\mu_{\mathrm{V}}\pm\theta)(\delta+\mu_{\mathrm{V}}\pm\theta)},$$

Let,

$$\mathrm{P} = \mu_{\mathrm{P}}\frac{(\mathrm{S_P}-S_P^*)^2}{\mathrm{S_P}} + \{1+\frac{\mu_{\mathrm{P}}}{\tau}\}\mu_{\mathrm{P}}\mathrm{I_P}\} + \frac{\mathrm{a}\beta_1 K^2\pi N_V^*(\mathrm{E_P}+\eta\mathrm{I_P})\mathrm{S_P}}{(\mathrm{k}\lambda_{\mathrm{V}}+\delta+\mu_{\mathrm{V}}\pm\theta)(\delta+\mu_{\mathrm{V}}\pm\theta)}$$
$$+\left(\frac{a^2\beta_1\beta_2 K^2\pi N_V^*(E_P^*+\eta I_P^*)}{(\mathrm{ka}\beta_2(E_P^*+\eta I_P^*)+\delta+\mu_{\mathrm{V}}\pm\theta)(\delta+\mu_{\mathrm{V}}\pm\theta)}\right)S_P^*\frac{S_P^*}{\mathrm{S_P}} + \omega\frac{S_P^*}{\mathrm{S_P}}\mathrm{E_P} + \{1+\frac{\mu_{\mathrm{P}}}{\tau}\}\tau\frac{I_P^*\mathrm{E_P}}{\mathrm{I_P}},$$

$$Q = \frac{a^2\beta_1\beta_2 K^2\pi N_V^* S_P^*}{(\mathrm{ka}\beta_2(\mathrm{E_P}+\eta\mathrm{I_P})+\delta+\mu_{\mathrm{V}}\pm\theta)(\delta+\mu_{\mathrm{V}}\pm\theta)}\{\mathrm{E_P}+\eta\mathrm{I_P}\frac{\mathrm{S_P}E_P^*}{\mathrm{E_P}}+\omega\frac{S_P^*E_P^*}{\mathrm{S_P}}+\{1+\frac{\mu_{\mathrm{P}}}{\tau}\}\mu_{\mathrm{P}}I_P^*+(\tau+\mu_{\mathrm{P}})E_P^*$$
$$+\frac{a^2\beta_1\beta_2 K^2\pi N_V^*(E_P^*+\eta I_P^*)S_P^*}{(\mathrm{ka}\beta_2(E_P^*+\eta I_P^*)+\delta+\mu_{\mathrm{V}}\pm\theta)(\delta+\mu_{\mathrm{V}}\pm\theta)},$$

$$\frac{\mathrm{dK}(\mathrm{S_P},\mathrm{E_P},\mathrm{I_P})}{\mathrm{dt}} = -P+Q,$$

The conditions necessary and sufficient for the EEP to be globally asymptotically stable in Lyapunov sense that is $\frac{\mathrm{dK}(\mathrm{S_P},\mathrm{E_P},\mathrm{I_P})}{\mathrm{dt}} \le 0$ is $P > Q$. This completes the proof.

3.9 Sensitivity analysis

The analytical sensitivity analysis of $\mathrm{R_0}$ with various parameters was obtained by partial differentiation as follows,

$$\frac{d\mathrm{R_0}}{d\omega} = -\frac{a^2\beta_1\beta_2 K^2\pi N_V^*\mathrm{N_P}}{(\delta+\mu_{\mathrm{V}}\pm\theta)^2(\omega+\tau+\mu_{\mathrm{P}})^2}\left\{1+\frac{\eta\tau}{\mu_{\mathrm{P}}}\right\} < 0;\ \frac{d\mathrm{R_0}}{d\delta} = -\frac{2a^2\beta_1\beta_2 K^2\pi N_V^*\mathrm{N_P}}{(\delta+\mu_{\mathrm{V}}\pm\theta)^3(\omega+\tau+\mu_{\mathrm{P}})^2}\left\{1+\frac{\eta\tau}{\mu_{\mathrm{P}}}\right\} < 0$$

When $\theta > 0$, $\frac{d\mathrm{R_0}}{d\theta} = -\frac{a^2\beta_1\beta_2 K^2\pi N_V^*\mathrm{N_P}}{(\delta+\mu_{\mathrm{V}}+\theta)^2(\omega+\tau+\mu_{\mathrm{P}})^2}\left\{1+\frac{\eta\tau}{\mu_{\mathrm{P}}}\right\} < 0$ and when $\theta < 0$ then

$$\frac{d\mathrm{R_0}}{d\theta} = \frac{a^2\beta_1\beta_2 K^2\pi N_V^*\mathrm{N_P}}{(\delta+\mu_{\mathrm{V}}+\theta)^2(\omega+\tau+\mu_{\mathrm{P}})^2}\left\{1+\frac{\eta\tau}{\mu_{\mathrm{P}}}\right\} > 0;\ \frac{\mathrm{dR_0}}{\mathrm{d}\tau} = \frac{\mathrm{a}^2\beta_1\beta_2\mathrm{K}^2\pi\mathrm{N_V^*N_P}(\eta\omega+\mu_{\mathrm{P}}[\eta-1])}{\mu_{\mathrm{P}}(\delta+\mu_{\mathrm{V}}\pm\theta)^2(\omega+\tau+\mu_{\mathrm{P}})^2} > 0$$

$$\frac{d\mathrm{R_0}}{d\beta_1} = \frac{a^2\beta_2 K^2\pi N_V^*\mathrm{N_P}}{(\delta+\mu_{\mathrm{V}}\pm\theta)^2(\omega+\tau+\mu_{\mathrm{P}})}\left\{1+\frac{\eta\tau}{\mu_{\mathrm{P}}}\right\} > 0;\ \frac{d\mathrm{R_0}}{d\beta_2} = \frac{a^2\beta_1 K^2\pi N_V^*\mathrm{N_P}}{(\delta+\mu_{\mathrm{V}}\pm\theta)^2(\omega+\tau+\mu_{\mathrm{P}})}\left\{1+\frac{\eta\tau}{\mu_{\mathrm{P}}}\right\} > 0$$

$$\frac{d\mathrm{R_0}}{d\mathrm{K}} = 2\frac{a^2\beta_1\beta_2\mathrm{K}\pi N_V^*\mathrm{N_P}}{(\delta+\mu_{\mathrm{V}}\pm\theta)^2(\omega+\tau+\mu_{\mathrm{P}})}\left\{1+\frac{\eta\tau}{\mu_{\mathrm{P}}}\right\} > 0;\ \frac{d\mathrm{R_0}}{d\mathrm{a}} = 2\frac{\mathrm{a}\beta_1\beta_2 K^2\pi N_V^*\mathrm{N_P}}{(\delta+\mu_{\mathrm{V}}\pm\theta)^2(\omega+\tau+\mu_{\mathrm{P}})}\left\{1+\frac{\eta\tau}{\mu_{\mathrm{P}}}\right\} > 0$$

The parameters which are greater than one are directly proportional to R_0 while those that are less than zero are inversely proportional to R_0. Climatic factors therefore affect immigration or emigration of insect vectors, their biting rate, infection rates and induced death rates.

4 Results and Discussion

The expression for basic reproduction number(R_0) obtained in model analysis estimates the number of secondary exposed plants and/or infectious plant that would be obtained if one infectious plant or exposed plant is introduced into the completely susceptible plant population. If $R_0 > 1$, the disease would invade the plant population and reach endemic equilibrium point (EEP) and should worry stakeholders. If $R_0 < 1$, the disease would die out.If $R_0 = 1$, the number of infected plants would remain constant.

The DFEP was found to be locally stable whenever$R_0 < 1$, which means that any starting values are near DFEP, they would move toward DFE asymptotically over time. The conditions necessary for global stability of DFEP and EEP were obtained, inferring that if those conditions are satisfied, any values they take would move toward equilibrium points asymptotically over time.

5 Conclusions

In this paper, we formulated a non-linear deterministic model for plant vector borne diseases by incorporating temperature and precipitation under the biting rate and wind as an agent of immigration and emigration, obtained the conditions for positivity and boundedness, local and global stability of the solution. The expression of vector population at equilibrium point was obtained and incorporated in model to consider shorter life span of vector. The expression for the basic reproduction number(R_0) was obtained. The basic reproduction number(R_0) estimates the number of secondary infection caused when one infected individual is introduced in a completely susceptible population. This study carried out analytical sensitivity analysis to determine how various parameters in the model affects R_0. Estimated numerical results, normalized sensitivity analysis and simulations to validate the model will be in our next paper. This study developed a deterministic model , future studies can consider a stochastic model.

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Competing Interests

Authors have declared that no competing interests exist.

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Appendix 1: List of variables and parameters

Variables	Descriptions
$N_V(t)$	The total population of the vectors
$N_P(t)$	The total population of the plants
$S_P(t)$,	The population of susceptible plant
$E_P(t)$	The population of exposed plant
$I_P(t)$.	The population of infected plant
$S_V(t)$	The population of susceptible vectors
$I_V(t)$	The population of infected vectors
Parameters	**Descriptions**
π	The recruitment rate of vectors
μ_P	The recruitment rate of plants
ω	The rates at which $E_P(t)$ recover naturally to $S_P(t)$.
δ	The rate at which climate induced death occur in both $S_V(t)$ and $I_V(t)$
μ_P	The constant natural death rate in subclasses $S_P(t)$, $E_P(t)$ *and* $I_P(t)$
τ	The rate at which $E_P(t)$ progresses to $I_P(t)$.
μ_V	The constant natural death rate in subclasses $S_V(t)$ *and* $I_V(t)$
θ	The immigration and emigration rate of vector (this implies that the rate can be either be positive or negative).
β_1	The infection rates for plants
β_2	The infection rates for vectors
a	The vector biting rate
λ_P	The number of individual plants which become infected at given time(force of infection of plants)
λ_V	The number of individual plants which become infected at given time(force of infection of plants)
k	The rate of change of force of infection due to climatic factors. Assumption is $K \geq 1$

Appendix 2: Definition of terms

Term	Definition
Susceptible population	The populations who are free of infection but are at a risk of contracting the infection.
Exposed population	The populations that has contracted the infection but are at a lower risk of transmitting it and/or with no signs of infection
Infected population	The population with the disease causing pathogen and capable of transmitting the infection to other plants or vectors on contact
Basic reproduction number/ratio	The number of cases one generates on average over the course of infectious period in a completely susceptible population
Bifurcation	It occurs when a smooth change is made to parameter values causes the system to change its behavior
Disease free equilibrium point	It is infection free point
Endemic equilibrium point	The stationary point at which the disease has completely invaded the population

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