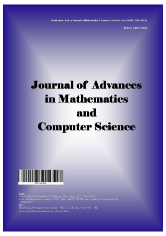

Journal of Advances in Mathematics and Computer Science

36(8): 137-154, 2021; Article no.JAMCS.73198

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

Unit Groups of Classes of Five Radical Zero Commutative Completely Primary Finite Rings

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Authors' contributions

This work was carried out in collaboration among all authors. Author HSW constructed the rings and determined the structure of the unit groups. Author MOO suggested the ideas on the construction of the rings and outlined the procedure for the determination of the unit groups while the author MNG managed the supervision of the manuscript. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JAMCS/2021/v36i830396

Editor(s):

(1) Dr. Dariusz Jacek Jakóbczak, Koszalin University of Technology, Poland.

Reviewers:

(1) Leonardo Simal Moreira, UniFoa Centro Universitário de Volta Redonda, Brazil.

(2) Mojgan Afkhami, University of Neyshabur, Iran.

Complete Peer review History: https://www.sdiarticle4.com/review-history/73198

Received: 10 August 2021

Accepted: 14 September 2021

Published: 16 October 2021

Original Research Article

Abstract

In this paper, R is considered a completely primary finite ring and $Z(R)$ is its subset of all zero divisors (including zero), forming a unique maximal ideal. We give a construction of R whose subset of zero divisors $Z(R)$ satisfies the conditions $(Z(R))^5 = (0); (Z(R))^4 \neq (0)$ and determine the structures of the unit groups of R for all its characteristics.

Keywords: Completely primary finite rings; five radical zero; unit groups.

2010 Mathematics Subject Classification: 53C25; 83C05; 57N16.

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1 Introduction

A comprehensive study on completely primary finite rings can be traced back to Raghavendran's publication [1]. Other related studies can be obtained from [2, 3, 4, 5, 6]. We shall denote the Jacobson radical of a completely primary finite ring R by $Z(R)$ while the rest of notations used in this paper are standard. The classification of finite rings is still inconclusive with some few expositions on the structures of unit groups and zero divisors of constructed rings. Chikunji in [7, 8] obtained the structures of group of units of classes of completely primary finite rings in which the product of any three zero divisors is zero. In [6], the authors determined the structure of the unit groups of completely primary finite rings in which the product of any four zero divisors is zero. We now construct a class of completely primary finite rings in which $(Z(R))^5 = (0)$ with $(Z(R))^4 \neq (0)$ and classify their group of units.

2 Preliminaries

The following are fundamental to the construction of a class of completely primary finite rings as well as classification of their unit groups in this paper.

a) A completely primary finite ring is a ring in which the set $Z(R)$ of all zero divisors forms a unique maximal ideal [2]. For more information on these rings, the reader is referred to [1].

b) Let R be a finite ring. Then there is no distinction between left and right zero divisors and every element is either a zero divisor or a unit [4, section 4].

c) Let R be a finite ring with multiplicative identity $1 \neq 0$, whose set of zero divisors form an additive group $Z(R)$. Then:

   (i) $Z(R)$ is the Jacobson radical of R;

   (ii) $|R| = p^{kr}$ and $|Z(R)| = p^{(k-1)r}$ for some prime p and some positive integers $k,\ r$;

   (iii) $(Z(R))^n = (0)$;

   (iv) The characteristic of the ring R is p^n for some integer n with $1 \leq n \leq k$ and if the characteristic is p^k , then R will be commutative.
   This is basically Theorem 2 of [1]

d) Let R be as in (c) above and let $CharR = p^k$. Then R has a coefficient subring $R_0 = GR(p^{kr}, p^k)$ with $CharR_0 = CharR$ and R_0/pR_0 equals to $R/Z(R)$. R_0 is clearly a maximal subring of R [3, Section 1].

e) Let R be a completely primary finite ring (not necessarily commutative). Then the group of units R^* of R contains a cyclic subgroup $< b >$ of order $p^r - 1$, and R^* is a semi direct product of $1 + Z(R)$ and $< b >$ [8, Proposition 2.1].

Remark 2.1. From (c) and (d) above, it is clear that if $(Z(R))^5 = (0)$ with $(Z(R))^4 \neq (0)$, then the characteristic of R is p^k, $1 \leq k \leq 5$.

3 Results

3.1 Construction of five radical zero commutative completely primary finite rings

Let $R_0 = GR(p^{kr}, p^k)$ be a Galois ring of order p^{kr} and characteristic p^k where p is a prime integer, $1 \leq k \leq 5$ and $r \in \mathbb{Z}^+$. Suppose U, V, W and Y are R_0/pR_0 - spaces considered as R_0 modules generated by e, f, g and h elements, respectively, such that the corresponding generating sets are

$\{u_1, \dots, u_e\}$, $\{v_1, \dots, v_f\}$, $\{w_1, \dots, w_g\}$ and $\{y_1, \dots, y_h\}$ so that $R = R_0 \bigoplus U \bigoplus V \bigoplus W \bigoplus Y$ is an additive abelian group. Then on the additive group, we define multiplication by the following relations:

(i) If $k = 1$, then

$$u_i u_{i'} = u_{i'} u_i = v_j, \quad u_i v_j = v_j u_i = w_k, \quad u_i w_k = w_k u_i = y_l, \quad u_i y_l = y_l u_i = 0,$$

$$v_j v_{j'} = v_{j'} v_j = y_l, \quad v_j w_k = w_k v_j = 0, \quad v_j y_l = y_l v_j = 0, \quad w_k w_{k'} = w_{k'} w_k = 0,$$

$$w_k y_l = y_l w_k = 0, \quad y_l y_{l'} = y_{l'} y_l = 0$$

(ii) If $k = 2$, then

$$u_i u_{i'} = u_{i'} u_i = pr_0 + pu_i + v_j, \; u_i v_j = v_j u_i = pu_i + w_k, u_i w_k = w_k u_i = pu_i + y_l,$$

$$u_i y_l = y_l u_i = pu_i, v_j v_{j'} = v_{j'} v_j = y_l, v_j w_k = w_k v_j = 0, v_j y_l = y_l v_j = 0, \; w_k w_{k'} = w_{k'} w_k = 0$$

$$w_k y_l = y_l w_k = 0, \quad y_l y_{l'} = y_{l'} y_l = 0$$

(iii) If $3 \leq k \leq 5$, then

$$u_i u_{i'} = u_{i'} u_i = p^2 r_0 + pu_i + v_j, \quad u_i v_j = v_j u_i = p^2 r_0 + pu_i + pv_j + w_k,$$

$$u_i w_k = w_k u_i = p^2 r_0 + pu_i + pw_k + y_l, \quad u_i y_l = y_l u_i = p^2 r_0 + pu_i,$$

$$v_j v_{j'} = v_{j'} v_j = p^2 r_0 + pv_j + y_l, \quad v_j w_k = w_k v_j = p^2 r_0 + pv_j + pw_k, v_j y_l = y_l v_j = p^2 r_0 + pv_j,$$

$$w_k w_{k'} = w_{k'} w_k = p^2 r_0 + pw_k, \quad w_k y_l = y_l w_k = p^2 r_0 + pw_k, \quad y_l y_{l'} = y_{l'} y_l = p^2 r_0.$$

Further $u_i u_{i'} u_{i''} u_{i'''} u_{i^{iv}} = 0$, $u_i r_0 = r_0 u_i$, $v_j r_0 = r_0 v_j$, $w_k r_0 = r_0 w_k, y_l r_0 = r_0 y_l$, where $r_0 \in R_0$ and $1 \leq i, \; i' \leq e, \; 1 \leq j, \; j' \leq f, \; 1 \leq k, \; k' \leq g, 1 \leq l, \; l' \leq h$. From the given multiplication in R, we see that if $r_0 + \sum_{i=1}^{e} r_i u_i + \sum_{j=1}^{f} s_j v_j + \sum_{t=1}^{g} t_k w_k + \sum_{l=1}^{h} z_l y_l$ and $r'_0 + \sum_{i=1}^{e} r'_i u_i + \sum_{j=1}^{f} s'_j v_j + \sum_{t=1}^{g} t'_k w_k + \sum_{l=1}^{h} z'_l y_l$ are any two elements of R, then

$$\left(r_0 + \sum_{i=1}^{e} r_i u_i + \sum_{j=1}^{f} s_j v_j + \sum_{k=1}^{g} t_k w_k + \sum_{l=1}^{h} z_l y_l\right)\left(r'_0 + \sum_{i=1}^{e} r'_i u_i + \sum_{j=1}^{f} s'_j v_j + \sum_{t=1}^{g} t'_k w_k + \sum_{l=1}^{h} z'_l y_l\right)$$

$$= r_0 r'_0 + p^a \sum_{i,m=1}^{e} \left(r_i r'_m + pR_0\right)$$

$$+ \sum_{i=1}^{e} \left[r_0 r'_i + r_i r'_0 + pR_0\right] u_i + \sum_{j=1}^{f} \left[(r_0 + pR_0)s'_j + s_j(r'_0 + pR_0) + \sum_{\nu,\ \mu=1}^{e} (r_\nu r'_\mu + pR_0)\right] v_j$$

$$+ \sum_{k=1}^{g} \left[(r_0 + pR_0)t'_k + t_k(r'_0 + pR_0) + \sum_{i,j}(r_i + pR_0)s'_j + s_j(r'_i + pR_0)\right] w_k$$

$$+ \sum_{l=1}^{h} \left[(r_0 + pR_0)z'_l + z_l(r'_0 + pR_0) + \sum_{i,k}(r_i + pR_0)t'_k + t_k(r'_i + pR_0) + \sum_{\kappa,\tau=1}^{f} (s_\kappa s'_\tau + pR_0)\right] y_l$$

where $a = 1, 2, 3,$ or 4 depending on whether $Char\ R_0 = p^2,\ p^3,\ p^4$ or p^5. It can be verified that this multiplication turns R into a commutative ring with identity 1.

Notice that if $R_0 = GR(p^r, p)$ where $Char\ R = p$, then the above multiplication reduces to

$$\left(r_0 + \sum_{i=1}^{e} r_i u_i + \sum_{j=1}^{f} s_j v_j + \sum_{k=1}^{g} t_k w_k + \sum_{l=1}^{h} z_l y_l\right)\left(r'_0 + \sum_{i=1}^{e} r'_i u_i + \sum_{j=1}^{f} s'_j v_j + \sum_{t=1}^{g} t'_k w_k + \sum_{l=1}^{h} z'_l y_l\right)$$

$$= r_0 r'_0 + \sum_{i=1}^{e} \left[r_0 r'_i + r_i r'_0\right] u_i + \sum_{j=1}^{f} \left[(r_0)s'_j + s_j(r'_0) + \sum_{\nu,\ \mu=1}^{e} (r_\nu r'_\mu)\right] v_j$$

$$+ \sum_{k=1}^{g} \left[(r_0)t'_k + t_k(r'_0) + \sum_{i,j} (r_i)s'_j + s_j(r'_i)\right] w_k$$

$$+ \sum_{l=1}^{h} \left[(r_0)z'_l + z_l(r'_0) + \sum_{i,k} (r_i)t'_k + t_k(r'_i) + \sum_{\kappa,\tau=1}^{f} (s_\kappa s'_\tau)\right] y_l$$

In the sequel, we use the ideas of Raghavendran [1] and Chikunji [8] to classify the unit groups of the rings constructed in this section. Evidently

$$\begin{aligned} Z(R) &= pR_0 \bigoplus U \bigoplus V \bigoplus W \bigoplus Y \\ &= pR_0 + \sum_{i=1}^{e} R_0 u_i + \sum_{j=1}^{f} R_0 v_j + \sum_{k=1}^{g} R_0 w_k + \sum_{l=1}^{h} R_0 y_l \end{aligned}$$

is a unique maximal ideal of R and

$$\begin{aligned} 1 + Z(R) &= 1 + pR_0 \bigoplus U \bigoplus V \bigoplus W \bigoplus Y \\ &= 1 + pR_0 + \sum_{i=1}^{e} R_0 u_i + \sum_{j=1}^{f} R_0 v_j + \sum_{k=1}^{g} R_0 w_k + \sum_{l=1}^{h} R_0 y_l \end{aligned}$$

and

$$R^* = (R^*/1 + Z(R)) \times (1 + Z(R)) =< b > \times (1 + Z(R))$$

where

$$< b >= (R^*/1 + Z(R)) = (R/Z(R))^* = \mathbb{F}^*_{p^r} \cong \mathbb{Z}_{p^r-1}$$

Proposition 3.1. *Let R be a completely primary finite ring from the class of finite rings described by the construction and of characteristic p with $pu_i = pv_j = pw_k = py_l = 0$. Then the group of units*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times (\mathbb{Z}_8^e)^r \times (\mathbb{Z}_2^g)^r, & \text{if } p = 2 \\ \mathbb{Z}_{3^r-1} \times (\mathbb{Z}_9^e)^r \times (\mathbb{Z}_3^f)^r \times (\mathbb{Z}_3^h)^r, & \text{if } p = 3 \\ \mathbb{Z}_{p^r-1} \times (\mathbb{Z}_p^e)^r \times (\mathbb{Z}_p^f)^r \times (\mathbb{Z}_p^g)^r \times (\mathbb{Z}_p^h)^r, & \text{if } p > 3 \end{cases}$$

Proof. Using the fact that $R^* \cong \mathbb{Z}_{p^r-1} \times (1 + Z(R))$, it suffices to determine the structure of $1 + Z(R)$. Let $\varepsilon_1, \ldots, \varepsilon_r$ be elements of R_0 with $\varepsilon_1 = 1$ such that $\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_r$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbb{F}_p$. We consider the three cases separately.
Case(i): $p = 2$. For every $t = 1, \ldots, r$, $(1 + \varepsilon_t u_i)^8 = 1$ and $(1 + \varepsilon_t w_k)^2 = 1$. For non-negative integers α_t and λ_t with $\alpha_t \leq 2$, and $\lambda_t \leq 8$, it is clear that

$$\prod_{i=1}^{e}\prod_{t=1}^{r} \left\{(1 + \varepsilon_t u_i)^{\lambda_t}\right\} \cdot \prod_{k=1}^{g}\prod_{t=1}^{r} \{(1 + \varepsilon_t w_k)^{\alpha_t}\} = \{1\}$$

This indicates that $\lambda_t = 8$ and $\alpha_t = 2$ for all $t = 1, \ldots, r$.
Suppose
$A_{ti} = \left\{(1+\varepsilon_t u_i)^{\lambda} : \lambda = 1, \ldots, 8; \forall t = 1, \ldots, r\right\}$ and
$B_{tk} = \{(1+\varepsilon_t w_k)^{\alpha} : \alpha = 1, 2; \forall t = 1, \ldots, r\}$,
then A_{ti} and B_{tk} are all cyclic subgroups of the group $1 + Z(R)$ and these are of the orders inferred from their definition. Since the intersection of the cyclic subgroups $< 1+\varepsilon_t u_i >$ and $< 1+\varepsilon_t w_k >$ gives the identity group and that

$$\left|\prod_{i=1}^{e}\prod_{t=1}^{r} < 1+\varepsilon_t u_i >\right| \cdot \left|\prod_{k=1}^{g}\prod_{t=1}^{r} < 1+\varepsilon_t w_k >\right|$$

coincides with $|1 + Z(R)|$, it follows that

$$\begin{aligned} 1 + Z(R) &= \prod_{i=1}^{e}\prod_{t=1}^{r} < 1+\varepsilon_t u_i > \times \prod_{k=1}^{g}\prod_{t=1}^{r} < 1+\varepsilon_t w_k > \\ &\cong (\mathbb{Z}_8^e)^r \times (\mathbb{Z}_2^g)^r \end{aligned}$$

Case(ii): $p = 3$. For every $t = 1, \ldots, r$, $(1+\varepsilon_t u_i)^9 = 1$, $(1+\varepsilon_t v_j)^3 = 1$ and $(1+\varepsilon_t y_l)^3 = 1$. For non-negative integers λ_t, α_t and φ_t with $\lambda_t \le 9$, $\alpha_t \le 3$ and $\varphi_t \le 3$, it is clear that

$$\prod_{i=1}^{e}\prod_{t=1}^{r}\left\{(1+\varepsilon_t u_i)^{\lambda_t}\right\} \cdot \prod_{j=1}^{f}\prod_{t=1}^{r}\{(1+\varepsilon_t v_j)^{\alpha_t}\} \cdot \prod_{l=1}^{h}\prod_{t=1}^{r}\{(1+\varepsilon_t y_l)^{\varphi_t}\} = \{1\}$$

This indicates that $\lambda_t = 9, \alpha_t = 3$ and $\varphi_t = 3$ for all $t = 1, \ldots, r$.
Suppose
$A_{ti} = \left\{(1+\varepsilon_t u_i)^{\lambda} : \lambda = 1, \ldots, 9; \forall t = 1, \ldots, r\right\}$,
$B_{tj} = \{(1+\varepsilon_t v_j)^{\alpha} : \alpha = 1, 2, 3; \forall t = 1, \ldots, r\}$, and
$C_{tl} = \{(1+\varepsilon_t y_l)^{\varphi} : \varphi = 1, 2, 3; \forall t = 1, \ldots, r\}$,
then A_{ti}, B_{tj} and C_{tl} are all cyclic subgroups of the group $1 + Z(R)$ and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups $< 1+\varepsilon_t u_i >$, $< 1+\varepsilon_t v_j >$, and $< 1+\varepsilon_t y_l >$ gives the identity group and that

$$\left|\prod_{i=1}^{e}\prod_{t=1}^{r} < 1+\varepsilon_t u_i >\right| \cdot \left|\prod_{j=1}^{f}\prod_{t=1}^{r} < 1+\varepsilon_t v_j >\right| \cdot \left|\prod_{l=1}^{h}\prod_{t=1}^{r} < 1+\varepsilon_t y_l >\right|$$

coincides with $|1 + Z(R)|$, it follows that

$$\begin{aligned} 1 + Z(R) &= \prod_{i=1}^{e}\prod_{t=1}^{r} < 1+\varepsilon_t u_i > \times \prod_{j=1}^{f}\prod_{t=1}^{r} < 1+\varepsilon_t v_j > \times \prod_{l=1}^{h}\prod_{t=1}^{r} < 1+\varepsilon_t y_l > \\ &\cong (\mathbb{Z}_9^e)^r \times (\mathbb{Z}_3^f)^r \times (\mathbb{Z}_3^h)^r \end{aligned}$$

Case(iii): $p > 3$. For every $t = 1, \ldots, r$, $(1+\varepsilon_t u_i)^p = 1$, $(1+\varepsilon_t v_j)^p = 1$, $(1+\varepsilon_t w_k)^p = 1$, and $(1+\varepsilon_t y_l)^p = 1$. For non-negative integers α_t, φ_t, δ_t and λ_t with $\alpha_t \le p$, $\varphi_t \le p$, $\delta_t \le p$ and $\lambda_t \le p$, it is clear that

$$\prod_{i=1}^{e}\prod_{t=1}^{r}\{(1+\varepsilon_t u_i)^{\alpha_t}\} \cdot \prod_{j=1}^{f}\prod_{t=1}^{r}\{(1+\varepsilon_t v_j)^{\varphi_t}\} \cdot \prod_{k=1}^{g}\prod_{t=1}^{r}\left\{(1+\varepsilon_t w_k)^{\delta_t}\right\} \cdot \prod_{l=1}^{h}\prod_{t=1}^{r}\left\{(1+\varepsilon_t y_l)^{\lambda_t}\right\} = \{1\}$$

This indicates that $\alpha_t = p$, $\varphi_t = p$, $\delta_t = p$ and $\lambda_t = p$ for all $t = 1, \ldots, r$.
Suppose

$A_{ti} = \{(1+\varepsilon_t u_i)^{\alpha} : \alpha = 1, \ldots, p; \forall t = 1, \ldots, r\}$,
$B_{tj} = \{(1+\varepsilon_t v_j)^{\varphi} : \varphi = 1, \ldots, p; \forall t = 1, \ldots, r\}$,
$C_{tk} = \left\{(1+\varepsilon_t w_k)^{\delta} : \delta = 1, \ldots, p; \forall t = 1, \ldots, r\right\}$, and
$D_{tl} = \left\{(1+\varepsilon_t y_l)^{\lambda} : \lambda = 1, \ldots, p; \forall t = 1, \ldots, r\right\}$,
then A_{ti}, B_{tj}, C_{tk} and D_{tl} are all cyclic subgroups of the group $1 + Z(R)$ and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups $< 1+\varepsilon_t u_i >$, $< 1+\varepsilon_t v_j >$, $< 1+\varepsilon_t w_k >$ and $< 1+\varepsilon_t y_l >$ gives the identity group and that

$$\left|\prod_{i=1}^{e}\prod_{t=1}^{r} < 1+\varepsilon_t u_i >\right| \cdot \left|\prod_{j=1}^{f}\prod_{t=1}^{r} < 1+\varepsilon_t v_j >\right| \cdot \left|\prod_{k=1}^{g}\prod_{t=1}^{r} < 1+\varepsilon_t w_k >\right| \cdot \left|\prod_{l=1}^{h}\prod_{t=1}^{r} < 1+\varepsilon_t y_l >\right|$$

coincides with $|1 + Z(R)|$, it follows that

$$\begin{aligned}1 + Z(R) &= \prod_{i=1}^{e}\prod_{t=1}^{r} < 1+\varepsilon_t u_i > \times \prod_{j=1}^{f}\prod_{t=1}^{r} < 1+\varepsilon_t v_j > \times \prod_{k=1}^{g}\prod_{t=1}^{r} < 1+\varepsilon_t w_k > \times \prod_{l=1}^{h}\prod_{t=1}^{r} < 1+\varepsilon_t y_l > \\ &\cong (\mathbb{Z}_p^e)^r \times (\mathbb{Z}_p^f)^r \times (\mathbb{Z}_p^g)^r \times (\mathbb{Z}_p^h)^r\end{aligned}$$

□

Proposition 3.2. *Let R be a completely primary finite ring from the class of finite rings described by the construction and of characteristic p^2 with $pu_i = pv_j = pw_k = py_l$. Then the group of units*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_2^r \times (\mathbb{Z}_8^e)^r \times (\mathbb{Z}_2^g)^r, & \text{if } p = 2 \\ \mathbb{Z}_{3^r-1} \times \mathbb{Z}_3^r \times (\mathbb{Z}_9^e)^r \times (\mathbb{Z}_3^f)^r \times (\mathbb{Z}_3^h)^r, & \text{if } p = 3 \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_p^r \times (\mathbb{Z}_p^e)^r \times (\mathbb{Z}_p^f)^r \times (\mathbb{Z}_p^g)^r \times (\mathbb{Z}_p^h)^r, & \text{if } p > 3 \end{cases}$$

Proof. Using the fact that $R^* \cong \mathbb{Z}_{p^r-1} \times (1 + Z(R))$, it suffices to determine the structure of $1+Z(R)$. Let $\varepsilon_1, \ldots, \varepsilon_r$ be elements of R_0 with $\varepsilon_1 = 1$ such that $\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_r$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbb{F}_p$. We consider the three cases separately:
Case (i): $p = 2$. For every $t = 1, \ldots, r$, $(1 + 2\varepsilon_t)^2 = 1$, $(1 + \varepsilon_t u_i)^8 = 1$, and $(1 + \varepsilon_t w_k)^2 = 1$. For non-negative integers α_t, λ_t and δ_t with $\alpha_t \leq 2$, $\lambda_t \leq 8$ and $\delta_t \leq 2$, it is clear that

$$\prod_{t=1}^{r}\{(1+2\varepsilon_t)^{\alpha_t}\} \cdot \prod_{i=1}^{e}\prod_{t=1}^{r}\left\{(1+\varepsilon_t u_i)^{\lambda_t}\right\} \cdot \prod_{k=1}^{g}\prod_{t=1}^{r}\left\{(1+\varepsilon_t w_k)^{\delta_t}\right\} = \{1\}$$

This indicates that $\alpha_t = 2$, $\lambda_t = 8$ and $\delta_t = 2$ for all $t = 1, \ldots, r$.

Suppose
$A_t = \{(1+2\varepsilon_t)^{\alpha} : \alpha = 1, 2; \forall t = 1, \ldots, r\}$,
$B_{ti} = \left\{(1+\varepsilon_t u_i)^{\lambda} : \lambda = 1, \ldots, 8; \forall t = 1, \ldots, r\right\}$, and
$C_{tk} = \left\{(1+\varepsilon_t w_k)^{\delta} : \delta = 1, 2; \forall t = 1, \ldots, r\right\}$,
then A_t, B_{ti} and C_{tk} are all cyclic subgroups of the group $1 + Z(R)$ and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups $< 1+2\varepsilon_t >$, $< 1+\varepsilon_t u_i >$, and $< 1+\varepsilon_t w_k >$ gives the identity group and that

$$\left|\prod_{t=1}^{r} < 1+2\varepsilon_t >\right| \cdot \left|\prod_{i=1}^{e}\prod_{t=1}^{r} < 1+\varepsilon_t u_i >\right| \cdot \left|\prod_{k=1}^{g}\prod_{t=1}^{r} < 1+\varepsilon_t w_k >\right|$$

coincides with $|1+Z(R)|$, it follows that

$$1+Z(R)=\prod_{t=1}^{r}<1+2\varepsilon_t>\times\prod_{i=1}^{e}\prod_{t=1}^{r}<1+\varepsilon_t u_i>\times\prod_{k=1}^{g}\prod_{t=1}^{r}<1+\varepsilon_t w_k>$$
$$\cong\mathbb{Z}_2^r\times(\mathbb{Z}_8^e)^r\times(\mathbb{Z}_2^g)^r$$

Case(ii): $p=3$. For every $t=1,\ldots,r$, $(1+3\varepsilon_t)^3=1$, $(1+\varepsilon_t u_i)^9=1$, $(1+\varepsilon_t v_j)^3=1$, and $(1+\varepsilon_t y_l)^3=1$. For non-negative integers α_t, φ_t, δ_t and λ_t with $\alpha_t\le 3$, $\varphi_t\le 3$, $\delta_t\le 3$ and $\lambda_t\le 9$, it is clear that

$$\prod_{t=1}^{r}\left\{(1+3\varepsilon_t)^{\delta_t}\right\}\cdot\prod_{i=1}^{e}\prod_{t=1}^{r}\left\{(1+\varepsilon_t u_i)^{\lambda_t}\right\}\cdot\prod_{j=1}^{f}\prod_{t=1}^{r}\{(1+\varepsilon_t v_j)^{\varphi_t}\}\cdot\prod_{l=1}^{h}\prod_{t=1}^{r}\{(1+\varepsilon_t y_l)^{\alpha_t}\}=\{1\}$$

This indicates that $\alpha_t=3$, $\lambda_t=9$, $\delta_t=3$ and $\varphi_t=3$ for all $t=1,\ldots,r$.
Suppose
$A_t=\left\{(1+3\varepsilon_t)^{\delta}:\delta=1,2,3;\forall t=1,\ldots,r\right\}$,
$B_{ti}=\left\{(1+\varepsilon_t u_i)^{\lambda}:\lambda=1,\ldots,9;\forall t=1,\ldots,r\right\}$,
$C_{tj}=\{(1+\varepsilon_t v_j)^{\varphi}:\varphi=1,2,3;\forall t=1,\ldots,r\}$, and
$D_{tl}=\{(1+\varepsilon_t y_l)^{\alpha}:\alpha=1,2,3;\forall t=1,\ldots,r\}$,
then A_t, B_{ti}, C_{tj} and D_{tl} are all cyclic subgroups of the group $1+Z(R)$ and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups $<1+3\varepsilon_t>$, $<1+\varepsilon_t u_i>$, $<1+\varepsilon_t v_j>$ and $<1+\varepsilon_t y_l>$ gives the identity group and that

$$\left|\prod_{t=1}^{r}<1+3\varepsilon_t>\right|\cdot\left|\prod_{i=1}^{e}\prod_{t=1}^{r}<1+\varepsilon_t u_i>\right|\cdot\left|\prod_{j=1}^{f}\prod_{t=1}^{r}<1+\varepsilon_t v_j>\right|\cdot\left|\prod_{l=1}^{h}\prod_{t=1}^{r}<1+\varepsilon_t y_l>\right|$$

coincides with $|1+Z(R)|$, it follows that

$$1+Z(R)=\prod_{t=1}^{r}<1+3\varepsilon_t>\times\prod_{i=1}^{e}\prod_{t=1}^{r}<1+\varepsilon_t u_i>\times\prod_{j=1}^{f}\prod_{t=1}^{r}<1+\varepsilon_t v_j>\times\prod_{l=1}^{h}\prod_{t=1}^{r}<1+\varepsilon_t y_l>$$
$$\cong\mathbb{Z}_3^r\times(\mathbb{Z}_9^e)^r\times(\mathbb{Z}_3^f)^r\times(\mathbb{Z}_3^h)^r$$

Case(iii): $p>3$. For every $t=1,\ldots,r$, $(1+p\varepsilon_t)^p=1$, $(1+\varepsilon_t u_i)^p=1$, $(1+\varepsilon_t v_j)^p=1$, $(1+\varepsilon_t w_k)^p=1$, and $(1+\varepsilon_t y_l)^p=1$. For non-negative integers α_t, φ_t, δ_t, β_t and λ_t such that $\alpha_t\le p$, $\varphi_t\le p$, $\delta_t\le p$, $\beta_t\le p$ and $\lambda_t\le p$, it is clear that

$$\prod_{t=1}^{r}\{(1+p\varepsilon_t)^{\alpha_t}\}\cdot\prod_{i=1}^{e}\prod_{t=1}^{r}\{(1+\varepsilon_t u_i)^{\varphi_t}\}\cdot\prod_{j=1}^{f}\prod_{t=1}^{r}\left\{(1+\varepsilon_t v_j)^{\delta_t}\right\}\cdot\prod_{k=1}^{g}\prod_{t=1}^{r}\left\{(1+\varepsilon_t w_k)^{\beta_t}\right\}\cdot$$
$$\prod_{l=1}^{h}\prod_{t=1}^{r}\left\{(1+\varepsilon_t y_l)^{\lambda_t}\right\}=\{1\}$$

This indicates that $\alpha_t=p$, $\lambda_t=p$, $\delta_t=p$, $\beta_t=p$ and $\varphi_t=p$ for all $t=1,\ldots,r$.
Suppose
$A_t=\{(1+p\varepsilon_t)^{\alpha}:\alpha=1,\ldots,p;\forall t=1,\ldots,r\}$,
$B_{ti}=\{(1+\varepsilon_t u_i)^{\varphi}:\varphi=1,\ldots,p;\forall t=1,\ldots,r\}$,
$C_{tj}=\left\{(1+\varepsilon_t v_j)^{\delta}:\delta=1,\ldots,p;\forall t=1,\ldots,r\right\}$,
$D_{tk}=\left\{(1+\varepsilon_t w_k)^{\beta}:\beta=1,\ldots,p;\forall t=1,\ldots,r\right\}$, and
$E_{tl}=\left\{(1+\varepsilon_t y_l)^{\lambda}:\lambda=1,\ldots,p;\forall t=1,\ldots,r\right\}$,

then A_t, B_{ti}, C_{tj}, D_{tk} and E_{tl} are all cyclic subgroups of the group $1 + Z(R)$ and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups $< 1 + p\varepsilon_t >$, $< 1 + \varepsilon_t u_i >$, $< 1 + \varepsilon_t v_j >$, $< 1 + \varepsilon_t w_k >$ and $< 1 + \varepsilon_t y_l >$ gives the identity group and that

$$\left|\prod_{t=1}^{r} < 1 + p\varepsilon_t >\right| \cdot \left|\prod_{i=1}^{e}\prod_{t=1}^{r} < 1 + \varepsilon_t u_i >\right| \cdot \left|\prod_{j=1}^{f}\prod_{t=1}^{r} < 1 + \varepsilon_t v_j >\right| \cdot \left|\prod_{k=1}^{g}\prod_{t=1}^{r} < 1 + \varepsilon_t w_k >\right| \cdot \left|\prod_{l=1}^{h}\prod_{t=1}^{r} < 1 + \varepsilon_t y_l >\right|$$

coincides with $|1 + Z(R)|$, it follows that

$$\begin{aligned} 1 + Z(R) = &\prod_{t=1}^{r} < 1 + p\varepsilon_t > \times \prod_{i=1}^{e}\prod_{t=1}^{r} < 1 + \varepsilon_t u_i > \times \prod_{j=1}^{f}\prod_{t=1}^{r} < 1 + \varepsilon_t v_j > \times \prod_{k=1}^{g}\prod_{t=1}^{r} < 1 + \varepsilon_t w_k > \times \\ &\prod_{l=1}^{h}\prod_{t=1}^{r} < 1 + \varepsilon_t y_l > \\ &\cong \mathbb{Z}_p^r \times (\mathbb{Z}_p^e)^r \times (\mathbb{Z}_p^f)^r \times (\mathbb{Z}_p^g)^r \times (\mathbb{Z}_p^h)^r \end{aligned}$$

□

Proposition 3.3. *Let R be a completely primary finite ring from the class of finite rings described by the construction and of characteristic p^2 with $p^2u_i = pv_j = pw_k = py_l = 0$. Then the group of units*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_2^r \times (\mathbb{Z}_2^e)^r \times (\mathbb{Z}_2^g)^r \times (\mathbb{Z}_8^{e+f})^r, & \text{if } p = 2 \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_p^r \times (\mathbb{Z}_{p^2}^e)^r \times (\mathbb{Z}_p^f)^r \times (\mathbb{Z}_p^g)^r \times (\mathbb{Z}_p^h)^r, & \text{if } p \neq 2 \end{cases}$$

Proof. Since $R^* \cong \mathbb{Z}_{p^r-1} \times (1 + Z(R))$, it suffices to determine the structure of $1 + Z(R)$. Let $\varepsilon_1, \ldots, \varepsilon_r$ be elements of R_0 with $\varepsilon_1 = 1$ such that $\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_r$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbb{F}_p$. Then the generators with their respective orders are as indicated below:

Case(i): For $p = 2$, $1 \leq t \leq r$, $1 \leq i \leq e$, $1 \leq j \leq f$, $1 \leq k \leq g$, the generators are $1 + 2\varepsilon_t$ of order 2; $1 + 2\varepsilon_t u_i$ of order 2; $1 + \varepsilon_t w_k$ of order 2 and $1 + \varepsilon_t u_i + \varepsilon_t v_j$ of order 8. The rest of the proof is similar to the proof of Proposition 3.2.

Case(ii): For $p \neq 2$, $1 \leq t \leq r$, $1 \leq i \leq e$, $1 \leq j \leq f$, $1 \leq k \leq g$, $1 \leq l \leq h$, the generators are $1 + p\varepsilon_t$ of order p, $1 + \varepsilon_t u_i$ of order p^2, $1 + \varepsilon_t v_j$ of order p, $1 + \varepsilon_t w_k$ of order p and $1 + \varepsilon_t y_l$ of order p. The rest of the proof is similar to the proof of Proposition 3.2. □

Proposition 3.4. *Let R be a completely primary finite ring from the class of finite rings described by the construction and of characteristic p^3 with $pu_i = pv_j = pw_k = py_l = 0$. Then the group of units*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_2^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_8^e)^r \times (\mathbb{Z}_2^g)^r, & \text{if } p = 2 \\ \mathbb{Z}_{3^r-1} \times \mathbb{Z}_9^r \times (\mathbb{Z}_9^e)^r \times (\mathbb{Z}_3^f)^r \times (\mathbb{Z}_3^h)^r, & \text{if } p = 3 \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_p^r \times (\mathbb{Z}_p^e)^r \times (\mathbb{Z}_p^f)^r \times (\mathbb{Z}_p^g)^r \times (\mathbb{Z}_p^h)^r \times (\mathbb{Z}_p^{e+f})^r, & \text{if } p > 3 \end{cases}$$

Proof. Using the fact that $R^* \cong \mathbb{Z}_{p^r-1} \times (1 + Z(R))$, it suffices to determine the structure of $1 + Z(R)$. Let $\varepsilon_1, \ldots, \varepsilon_r$ be elements of R_0 with $\varepsilon_1 = 1$ such that $\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_r$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbb{F}_p$. We consider three cases separately:

Case(i): $p = 2$. For every $t = 1, \dots, r$, $(1+2\varepsilon_t)^2 = 1$, $(1+4\varepsilon_t)^2 = 1$, $(1+\varepsilon_t u_i)^8 = 1$, and $(1+\varepsilon_t w_k)^2 = 1$. For non-negative integers α_t, δ_t, φ_t and λ_t with $\alpha_t \le 2$, $\delta_t \le 2$, $\varphi_t \le 8$ and $\lambda_t \le 2$, it is clear that

$$\prod_{t=1}^{r}\{(1+2\varepsilon_t)^{\alpha_t}\} \cdot \prod_{t=1}^{r}\left\{(1+4\varepsilon_t)^{\delta_t}\right\} \cdot \prod_{i=1}^{e}\prod_{t=1}^{r}\{(1+\varepsilon_t u_i)^{\varphi_t}\} \cdot \prod_{k=1}^{g}\prod_{t=1}^{r}\left\{(1+\varepsilon_t w_k)^{\lambda_t}\right\} = \{1\}$$

This indicates that $\alpha_t = 2$, $\delta_t = 2$, $\varphi_t = 8$ and $\lambda_t = 2$ for all $t = 1, \dots, r$.
Suppose
$A_t = \{(1+2\varepsilon_t)^{\alpha} : \alpha = 1, 2; \forall t = 1, \dots, r\}$,
$B_t = \left\{(1+4\varepsilon_t)^{\delta} : \delta = 1, 2; \forall t = 1, \dots, r\right\}$,
$C_{ti} = \{(1+\varepsilon_t u_i)^{\varphi} : \varphi = 1, \dots, 8; \forall t = 1, \dots, r\}$, and
$D_{tk} = \left\{(1+\varepsilon_t w_k)^{\lambda} : \lambda = 1, 2; \forall t = 1, \dots, r\right\}$,
then A_t, B_t, C_{ti} and D_{tk} are all cyclic subgroups of the group $1 + Z(R)$ and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups $< 1+2\varepsilon_t >$, $< 1+4\varepsilon_t >$, $< 1+\varepsilon_t u_i >$ and $< 1+\varepsilon_t w_k >$ gives the identity group and that

$$\left|\prod_{t=1}^{r} < 1+2\varepsilon_t >\right| \cdot \left|\prod_{t=1}^{r} < 1+4\varepsilon_t >\right| \cdot \left|\prod_{i=1}^{e}\prod_{t=1}^{r} < 1+\varepsilon_t u_i >\right| \cdot \left|\prod_{k=1}^{g}\prod_{t=1}^{r} < 1+\varepsilon_t w_k >\right|$$

coincides with $|1 + Z(R)|$, it follows that

$$\begin{aligned} 1 + Z(R) &= \prod_{t=1}^{r} < 1+2\varepsilon_t > \times \prod_{t=1}^{r} < 1+4\varepsilon_t > \times \prod_{i=1}^{e}\prod_{t=1}^{r} < 1+\varepsilon_t u_i > \times \prod_{k=1}^{g}\prod_{t=1}^{r} < 1+\varepsilon_t w_k > \\ &\cong \mathbb{Z}_2^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_8^e)^r \times (\mathbb{Z}_2^g)^r \end{aligned}$$

Case(ii): $p = 3$. For every $t = 1, \dots, r$, $(1+6\varepsilon_t)^9 = 1$, $(1+\varepsilon_t u_i)^9 = 1$, $(1+\varepsilon_t v_j)^3 = 1$, and $(1+\varepsilon_t y_l)^3 = 1$. For non-negative integers α_t, λ_t, φ_t and δ_t with $\alpha_t \le 3$, $\lambda_t \le 9$, $\varphi_t \le 3$ and $\delta_t \le 9$, it is clear that

$$\prod_{t=1}^{r}\left\{(1+6\varepsilon_t)^{\delta_t}\right\} \cdot \prod_{i=1}^{e}\prod_{t=1}^{r}\left\{(1+\varepsilon_t u_i)^{\lambda_t}\right\} \cdot \prod_{j=1}^{f}\prod_{t=1}^{r}\{(1+\varepsilon_t v_j)^{\varphi_t}\} \cdot \prod_{l=1}^{h}\prod_{t=1}^{r}\{(1+\varepsilon_t y_l)^{\alpha_t}\} = \{1\}$$

This indicates that $\delta_t = 9$, $\lambda_t = 9$, $\varphi_t = 3$ and $\alpha_t = 3$ for all $t = 1, \dots, r$.
Suppose
$A_t = \left\{(1+6\varepsilon_t)^{\delta} : \delta = 1, \dots, 9; \forall t = 1, \dots, r\right\}$,
$B_{ti} = \left\{(1+\varepsilon_t u_i)^{\lambda} : \lambda = 1, \dots, 9; \forall t = 1, \dots, r\right\}$,
$C_{tj} = \left\{(1+\varepsilon_t v_j)^{\varphi} : \varphi = 1, 2, 3; \forall t = 1, \dots, r\right\}$, and
$D_{tl} = \{(1+\varepsilon_t y_l)^{\alpha} : \alpha = 1, 2, 3; \forall t = 1, \dots, r\}$,
then A_t, B_{ti}, C_{tj} and D_{tl} are all cyclic subgroups of the group $1 + Z(R)$ and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups $< 1+6\varepsilon_t >$, $< 1+\varepsilon_t u_i >$, $< 1+\varepsilon_t v_j >$ and $< 1+\varepsilon_t y_l >$ gives the identity group and that

$$\left|\prod_{t=1}^{r} < 1+6\varepsilon_t >\right| \cdot \left|\prod_{i=1}^{e}\prod_{t=1}^{r} < 1+\varepsilon_t u_i >\right| \cdot \left|\prod_{j=1}^{f}\prod_{t=1}^{r} < 1+\varepsilon_t v_j >\right| \cdot \left|\prod_{l=1}^{h}\prod_{t=1}^{r} < 1+\varepsilon_t y_l >\right|$$

coincides with $|1 + Z(R)|$, it follows that

$$\begin{aligned} 1 + Z(R) &= \prod_{t=1}^{r} < 1+6\varepsilon_t > \times \prod_{i=1}^{e}\prod_{t=1}^{r} < 1+\varepsilon_t u_i > \times \prod_{j=1}^{f}\prod_{t=1}^{r} < 1+\varepsilon_t v_j > \times \prod_{l=1}^{h}\prod_{t=1}^{r} < 1+\varepsilon_t y_l > \\ &\cong \mathbb{Z}_9^r \times (\mathbb{Z}_9^e)^r \times (\mathbb{Z}_3^f)^r \times (\mathbb{Z}_3^h)^r \end{aligned}$$

Case(iii): $p > 3$. For every $t = 1, \ldots, r$, $(1 + p^2\varepsilon_t)^p = 1$, $(1 + \varepsilon_t u_i)^p = 1$, $(1 + \varepsilon_t v_j)^p = 1$, $(1 + \varepsilon_t w_k)^p = 1$, $(1 + \varepsilon_t y_l)^p = 1$, and $(1 + \varepsilon_t u_i + \varepsilon_t v_j)^p = 1$. For non-negative integers α_t, φ_t, δ_t, β_t, η_t, and λ_t with $\alpha_t \le p$, $\varphi_t \le p$, $\delta_t \le p$, $\beta_t \le p$, $\eta_t \le p$, and $\lambda_t \le p$, it is clear that

$$\prod_{t=1}^{r}\left\{(1+p^2\varepsilon_t)^{\alpha_t}\right\} \cdot \prod_{i=1}^{e}\prod_{t=1}^{r}\left\{(1+\varepsilon_t u_i)^{\varphi_t}\right\} \cdot \prod_{j=1}^{f}\prod_{t=1}^{r}\left\{(1+\varepsilon_t v_j)^{\delta_t}\right\} \cdot \prod_{k=1}^{g}\prod_{t=1}^{r}\left\{(1+\varepsilon_t w_k)^{\beta_t}\right\} \cdot$$
$$\prod_{l=1}^{h}\prod_{t=1}^{r}\left\{(1+\varepsilon_t y_l)^{\eta_t}\right\} \cdot \prod_{j=1}^{f}\prod_{i=1}^{e}\prod_{t=1}^{r}\left\{(1+\varepsilon_t u_i + \varepsilon_t v_j)^{\lambda_t}\right\} = \{1\}$$

This indicates that $\alpha_t = p$, $\lambda_t = p$, $\varphi_t = p$, $\delta_t = p$, $\beta_t = p$, and $\eta_t = p$ for all $t = 1, \ldots, r$.
Suppose
$A_t = \left\{(1 + p^2\varepsilon_t)^\alpha : \alpha = 1, \ldots, p; \forall t = 1, \ldots, r\right\}$,
$B_{ti} = \left\{(1 + \varepsilon_t u_i)^\varphi : \varphi = 1, \ldots, p; \forall t = 1, \ldots, r\right\}$,
$C_{tj} = \left\{(1 + \varepsilon_t v_j)^\delta : \delta = 1, \ldots, p; \forall t = 1, \ldots, r\right\}$,
$D_{tk} = \left\{(1 + \varepsilon_t w_k)^\beta : \beta = 1, \ldots, p; \forall t = 1, \ldots, r\right\}$,
$E_{tl} = \left\{(1 + \varepsilon_t y_l)^\eta : \eta = 1, \ldots, p; \forall t = 1, \ldots, r\right\}$, and
$F_{tij} = \left\{(1 + \varepsilon_t u_i + \varepsilon_t v_j)^\lambda : \lambda = 1, \ldots, p; \forall t = 1, \ldots, r\right\}$,
then A_t, B_{ti}, C_{tj}, D_{tk}, E_{tl}, and F_{tij} are all cyclic subgroups of the group $1 + Z(R)$ and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups $< 1 + p^2\varepsilon_t >$, $< 1 + \varepsilon_t u_i >$, $< 1 + \varepsilon_t v_j >$, $< 1 + \varepsilon_t w_k >$, $< 1 + \varepsilon_t y_l >$, and $< 1 + \varepsilon_t u_i + \varepsilon_t v_j >$ gives the identity group and that

$$\left|\prod_{t=1}^{r} < 1+p^2\varepsilon_t >\right| \cdot \left|\prod_{i=1}^{e}\prod_{t=1}^{r} < 1+\varepsilon_t u_i >\right| \cdot \left|\prod_{j=1}^{f}\prod_{t=1}^{r} < 1+\varepsilon_t v_j >\right| \cdot \left|\prod_{k=1}^{g}\prod_{t=1}^{r} < 1+\varepsilon_t w_k >\right| \cdot$$
$$\left|\prod_{l=1}^{h}\prod_{t=1}^{r} < 1+\varepsilon_t y_l >\right| \cdot \left|\prod_{j=1}^{f}\prod_{i=1}^{e}\prod_{t=1}^{r} < 1+\varepsilon_t u_i + \varepsilon_t v_j >\right|$$

coincides with $|1 + Z(R)|$, it follows that

$$1+Z(R) = \prod_{t=1}^{r} < 1+p^2\varepsilon_t > \times \prod_{i=1}^{e}\prod_{t=1}^{r} < 1+\varepsilon_t u_i > \times \prod_{j=1}^{f}\prod_{t=1}^{r} < 1+\varepsilon_t v_j > \times \prod_{k=1}^{g}\prod_{t=1}^{r} < 1+\varepsilon_t w_k > \times$$
$$\prod_{l=1}^{h}\prod_{t=1}^{r} < 1+\varepsilon_t y_l > \times \prod_{j=1}^{f}\prod_{i=1}^{e}\prod_{t=1}^{r} < 1+\varepsilon_t u_i + \varepsilon_t v_j >$$
$$\cong \mathbb{Z}_p^r \times (\mathbb{Z}_p^e)^r \times (\mathbb{Z}_p^f)^r \times (\mathbb{Z}_p^g)^r \times (\mathbb{Z}_p^h)^r \times (\mathbb{Z}_p^{e+f})^r$$

□

Proposition 3.5. *Let R be a completely primary finite ring from the class of finite rings described by the construction and of characteristic p^3 with $p^2u_i = pv_j = pw_k = py_l = 0$. Then the group of units*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_2^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_2^e)^r \times (\mathbb{Z}_2^g)^r \times (\mathbb{Z}_8^{e+f})^r, & \text{if } p = 2 \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^2}^r \times (\mathbb{Z}_{p^2}^e)^r \times (\mathbb{Z}_p^f)^r \times (\mathbb{Z}_p^g)^r \times (\mathbb{Z}_p^h)^r, & \text{if } p \neq 2 \end{cases}$$

Proof. Since $R^* \cong \mathbb{Z}_{p^r-1} \times (1 + Z(R))$, it suffices to determine the structure of $1 + Z(R)$. Let $\varepsilon_1, \ldots, \varepsilon_r$ be elements of R_0 with $\varepsilon_1 = 1$ such that $\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_r$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbb{F}_p$. Then the generators with their respective orders are as indicated below:

Case(i): For $p = 2,\ 1 \leq t \leq r,\ 1 \leq i \leq e, 1 \leq j \leq f,\ 1 \leq k \leq g$, the generators are $1 + 2\varepsilon_t$ of order 2; $1 + 4\varepsilon_t$ of oder 2, $1 + 2\varepsilon_t u_i$ of order 2; $1 + \varepsilon_t w_k$ of order 2 and $1 + \varepsilon_t u_i + \varepsilon_t v_j$ of order 8. The rest of the proof is similar to the proof of proposition 3.4.

*Case(ii):*For $p \neq 2$, $1 \leq t \leq r,\ 1 \leq i \leq e, 1 \leq j \leq f, 1 \leq k \leq g, 1 \leq l \leq h$, the generators are $1+p\varepsilon_t$ of order p^2, $1+\varepsilon_t u_i$ of order p^2, $1+\varepsilon_t v_j$ of order p, $1+\varepsilon_t w_k$ of order p and $1+\varepsilon_t y_l$ of order p. The rest of the proof is similar to the proof of Proposition 3.4. □

Proposition 3.6. *Let R be a completely primary finite ring from the class of finite rings described by the construction and of characteristic p^3 with $p^3 u_i = pv_j = pw_k = py_l = 0$. Then the group of units*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_2^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_8^e)^r \times (\mathbb{Z}_4^f)^r \times (\mathbb{Z}_2^g)^r, & \text{if } p = 2 \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^2}^r \times (\mathbb{Z}_{p^3}^e)^r \times (\mathbb{Z}_p^f)^r \times (\mathbb{Z}_p^g)^r \times (\mathbb{Z}_p^h)^r, & \text{if } p \neq 2 \end{cases}$$

Proof. Since $R^* \cong \mathbb{Z}_{p^r-1} \times (1 + Z(R))$, it suffices to determine the structure of $1 + Z(R)$. Let $\varepsilon_1, \ldots, \varepsilon_r$ be elements of R_0 with $\varepsilon_1 = 1$ such that $\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_r$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbb{F}_p$. Then the generators with their respective orders are as indicated below:

Case(i): For $p = 2,\ 1 \leq t \leq r,\ 1 \leq i \leq e, 1 \leq j \leq f,\ 1 \leq k \leq g$, the generators are $1 + 2\varepsilon_t$ of order 2; $1 + 4\varepsilon_t$ of oder 2, $1 + \varepsilon_t u_i$ of order 8; $1 + \varepsilon_t v_j$ of order 4 and $1 + \varepsilon_t w_k$ of order 2. The rest of the proof is similar to the proof of Proposition 3.4.

Case(ii): For $p \neq 2,\ 1 \leq t \leq r,\ 1 \leq i \leq e,\ 1 \leq j \leq f,\ 1 \leq k \leq g,\ 1 \leq l \leq h$, the generators are $1+p\varepsilon_t$ of order p^2, $1+\varepsilon_t u_i$ of order p^3, $1+\varepsilon_t v_j$ of order p, $1+\varepsilon_t w_k$ of order p and $1+\varepsilon_t y_l$ of order p. The rest of the proof is similar to the proof of Proposition 3.4. □

Proposition 3.7. *Let R be a completely primary finite ring from the class of finite rings described by the construction and of characteristic p^4 with $pu_i = pv_j = pw_k = py_l = 0$. Then the group of units*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_4^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_8^e)^r \times (\mathbb{Z}_2^g)^r, & \text{if } p = 2 \\ \mathbb{Z}_{2^r-1} \times \mathbb{Z}_{27}^r \times (\mathbb{Z}_9^e)^r \times (\mathbb{Z}_3^f)^r \times (\mathbb{Z}_3^h)^r, & \text{if } p = 3 \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^3}^r \times (\mathbb{Z}_p^e)^r \times (\mathbb{Z}_p^f)^r \times (\mathbb{Z}_p^g)^r \times (\mathbb{Z}_p^h)^r, & \text{if } p > 3 \end{cases}$$

Proof. Using the fact that $R^* \cong \mathbb{Z}_{p^r-1} \times (1 + Z(R))$, it suffices to determine the structure of $1 + Z(R)$. Let $\varepsilon_1, \ldots, \varepsilon_r$ be elements of R_0 with $\varepsilon_1 = 1$ such that $\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_r$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbb{F}_p$. We consider the three cases separately:

Case(i): $p = 2$. For every $t = 1, \ldots, r$, $(1 + 2\varepsilon_t)^4 = 1$, $(1 + 6\varepsilon_t)^2 = 1$, $(1 + \varepsilon_t u_i)^8 = 1$, and $(1+\varepsilon_t w_k)^2 = 1$. For non-negative integers $\alpha_t,\ \delta_t,\ \varphi_t$ and λ_t with $\alpha_t \leq 4,\ \delta_t \leq 2,\ \varphi_t \leq 8$ and $\lambda_t \leq 2$, it is clear that

$$\prod_{t=1}^{r} \{(1 + 2\varepsilon_t)^{\alpha_t}\} \cdot \prod_{t=1}^{r} \left\{(1 + 6\varepsilon_t)^{\delta_t}\right\} \cdot \prod_{i=1}^{e}\prod_{t=1}^{r} \{(1 + \varepsilon_t u_i)^{\varphi_t}\} \cdot \prod_{k=1}^{g}\prod_{t=1}^{r} \left\{(1 + \varepsilon_t w_k)^{\lambda_t}\right\} = \{1\}$$

This indicates that $\alpha_t = 4,\ \delta_t = 2,\ \varphi_t = 8$ and $\lambda_t = 2$ for all $t = 1, \ldots, r$.
Suppose
$A_t = \{(1 + 2\varepsilon_t)^{\alpha} : \alpha = 1, 2, 3, 4; \forall t = 1, \ldots, r\}$,
$B_t = \left\{(1 + 6\varepsilon_t)^{\delta} : \delta = 1, 2; \forall t = 1, \ldots, r\right\}$,
$C_{ti} = \{(1 + \varepsilon_t u_i)^{\varphi} : \varphi = 1, \ldots, 8; \forall t = 1, \ldots, r\}$, and
$D_{tk} = \left\{(1 + \varepsilon_t w_k)^{\lambda} : \lambda = 1, 2; \forall t = 1, \ldots, r\right\}$,

then A_t, B_t, C_{ti} and D_{tk} are all cyclic subgroups of the group $1+Z(R)$ and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups $< 1+2\varepsilon_t >$, $< 1+6\varepsilon_t >$, $< 1+\varepsilon_t u_i >$, and $< 1+\varepsilon_t w_k >$ gives the identity group and that

$$\left|\prod_{t=1}^{r} < 1+2\varepsilon_t >\right| \cdot \left|\prod_{t=1}^{r} < 1+6\varepsilon_t >\right| \cdot \left|\prod_{i=1}^{e}\prod_{t=1}^{r} < 1+\varepsilon_t u_i >\right| \cdot \left|\prod_{k=1}^{g}\prod_{t=1}^{r} < 1+\varepsilon_t w_k >\right|$$

coincides with $|1+Z(R)|$, it follows that

$$\begin{aligned}1+Z(R) &= \prod_{t=1}^{r} < 1+2\varepsilon_t > \times \prod_{t=1}^{r} < 1+6\varepsilon_t > \times \prod_{i=1}^{e}\prod_{t=1}^{r} < 1+\varepsilon_t u_i > \times \prod_{k=1}^{g}\prod_{t=1}^{r} < 1+\varepsilon_t w_k > \\ &\cong \mathbb{Z}_4^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_8^e)^r \times (\mathbb{Z}_2^g)^r\end{aligned}$$

Case(ii): $p=3$. For every $t=1,\ldots,r$, $(1+3\varepsilon_t)^{27}=1$, $(1+\varepsilon_t u_i)^9=1$, $(1+\varepsilon_t v_j)^3=1$, and $(1+\varepsilon_t y_l)^3=1$. For non-negative integers α_t, φ_t, δ_t and λ_t with $\alpha_t \le 27$, $\varphi_t \le 9$, $\delta_t \le 3$, and $\lambda_t \le 3$, it is clear that

$$\prod_{t=1}^{r}\{(1+3\varepsilon_t)^{\alpha_t}\} \cdot \prod_{i=1}^{e}\prod_{t=1}^{r}\{(1+\varepsilon_t u_i)^{\varphi_t}\} \cdot \prod_{j=1}^{f}\prod_{t=1}^{r}\left\{(1+\varepsilon_t v_j)^{\delta_t}\right\} \cdot \prod_{l=1}^{h}\prod_{t=1}^{r}\left\{(1+\varepsilon_t y_l)^{\lambda_t}\right\} = \{1\}$$

This indicates that $\alpha_t = 27$, $\lambda_t = 3$, $\varphi_t = 9$ and $\delta_t = 3$ for all $t=1,\ldots,r$.
Suppose
$A_t = \{(1+3\varepsilon_t)^{\alpha} : \alpha = 1,\ldots,27; \forall t = 1,\ldots,r\}$,
$B_{ti} = \{(1+\varepsilon_t u_i)^{\varphi} : \varphi = 1,\ldots,9; \forall t = 1,\ldots,r\}$,
$C_{tj} = \left\{(1+\varepsilon_t v_j)^{\delta} : \delta = 1,2,3; \forall t = 1,\ldots,r\right\}$, and
$D_{tl} = \left\{(1+\varepsilon_t y_l)^{\lambda} : \lambda = 1,2,3; \forall t = 1,\ldots,r\right\}$,
then A_t, B_{ti}, C_{tj} and D_{tl} are all cyclic subgroups of the group $1+Z(R)$ and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups $< 1+3\varepsilon_t >$, $< 1+\varepsilon_t u_i >$, $< 1+\varepsilon_t v_j >$, and $< 1+\varepsilon_t y_l >$ gives the identity group and that

$$\left|\prod_{t=1}^{r} < 1+3\varepsilon_t >\right| \cdot \left|\prod_{i=1}^{e}\prod_{t=1}^{r} < 1+\varepsilon_t u_i >\right| \cdot \left|\prod_{j=1}^{f}\prod_{t=1}^{r} < 1+\varepsilon_t v_j >\right| \cdot \left|\prod_{l=1}^{h}\prod_{t=1}^{r} < 1+\varepsilon_t y_l >\right|$$

coincides with $|1+Z(R)|$, it follows that

$$\begin{aligned}1+Z(R) &= \prod_{t=1}^{r} < 1+3\varepsilon_t > \times \prod_{i=1}^{e}\prod_{t=1}^{r} < 1+\varepsilon_t u_i > \times \prod_{j=1}^{f}\prod_{t=1}^{r} < 1+\varepsilon_t v_j > \times \prod_{l=1}^{h}\prod_{t=1}^{r} < 1+\varepsilon_t y_l > \\ &\cong \mathbb{Z}_{27}^r \times (\mathbb{Z}_9^e)^r \times (\mathbb{Z}_3^f)^r \times (\mathbb{Z}_3^h)^r\end{aligned}$$

Case(iii): $p>3$. For every $t=1,\ldots,r$, $(1+p\varepsilon_t)^{p^3}=1$, $(1+\varepsilon_t u_i)^p=1$, $(1+\varepsilon_t v_j)^p=1$, $(1+\varepsilon_t w_k)^p=1$, and $(1+\varepsilon_t y_l)^p=1$. For non-negative integers α_t, λ_t, φ_t, β_t, and δ_t with $\alpha_t \le p^3$, $\lambda_t \le p$, $\varphi_t \le p$, $\delta_t \le p$, and $\beta_t \le p$, it is clear that

$$\begin{aligned}&\prod_{t=1}^{r}\{(1+p\varepsilon_t)^{\alpha_t}\} \cdot \prod_{i=1}^{e}\prod_{t=1}^{r}\{(1+\varepsilon_t u_i)^{\varphi_t}\} \cdot \prod_{j=1}^{f}\prod_{t=1}^{r}\left\{(1+\varepsilon_t v_j)^{\delta_t}\right\} \cdot \prod_{k=1}^{g}\prod_{t=1}^{r}\left\{(1+\varepsilon_t w_k)^{\beta_t}\right\} \cdot \\ &\prod_{l=1}^{h}\prod_{t=1}^{r}\left\{(1+\varepsilon_t y_l)^{\lambda_t}\right\} = \{1\}\end{aligned}$$

This indicates that $\alpha_t = p^3$, $\lambda_t = p$, $\varphi_t = p$, $\delta_t = p$, and $\beta_t = p$, for all $t=1,\ldots,r$.

Suppose
$A_t = \left\{(1+p\varepsilon_t)^{\alpha} : \alpha = 1,\ldots,p^3; \forall t = 1,\ldots,r\right\}$,
$B_{ti} = \{(1+\varepsilon_t u_i)^{\varphi} : \varphi = 1,\ldots,p; \forall t = 1,\ldots,r\}$,
$C_{tj} = \left\{(1+\varepsilon_t v_j)^{\delta} : \delta = 1,\ldots,p; \forall t = 1,\ldots,r\right\}$,
$D_{tk} = \left\{(1+\varepsilon_t w_k)^{\beta} : \beta = 1,\ldots,p; \forall t = 1,\ldots,r\right\}$, and
$E_{tl} = \left\{(1+\varepsilon_t y_l)^{\lambda} : \lambda = 1,\ldots,p; \forall t = 1,\ldots,r\right\}$
then A_t, B_{ti}, C_{tj}, D_{tk}, and E_{tl}, are all cyclic subgroups of the group $1+Z(R)$ and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups $< 1+p\varepsilon_t >$, $< 1+\varepsilon_t u_i >$, $< 1+\varepsilon_t v_j >$, $< 1+\varepsilon_t w_k >$, and $< 1+\varepsilon_t y_l >$, gives the identity group and that

$$\left|\prod_{t=1}^{r} < 1+p\varepsilon_t >\right| \cdot \left|\prod_{i=1}^{e}\prod_{t=1}^{r} < 1+\varepsilon_t u_i >\right| \cdot \left|\prod_{j=1}^{f}\prod_{t=1}^{r} < 1+\varepsilon_t v_j >\right| \cdot \left|\prod_{k=1}^{g}\prod_{t=1}^{r} < 1+\varepsilon_t w_k >\right| \cdot \left|\prod_{l=1}^{h}\prod_{t=1}^{r} < 1+\varepsilon_t y_l >\right|$$

coincides with $|1+Z(R)|$, it follows that

$$\begin{aligned} 1+Z(R) &= \prod_{t=1}^{r} < 1+p\varepsilon_t > \times \prod_{i=1}^{e}\prod_{t=1}^{r} < 1+\varepsilon_t u_i > \times \prod_{j=1}^{f}\prod_{t=1}^{r} < 1+\varepsilon_t v_j > \\ &\times \prod_{k=1}^{g}\prod_{t=1}^{r} < 1+\varepsilon_t w_k > \times \prod_{l=1}^{h}\prod_{t=1}^{r} < 1+\varepsilon_t y_l > \\ &\cong \mathbb{Z}_{p^3}^r \times (\mathbb{Z}_p^e)^r \times (\mathbb{Z}_p^f)^r \times (\mathbb{Z}_p^g)^r \times (\mathbb{Z}_p^h)^r \end{aligned}$$

□

Proposition 3.8. *Let R be a completely primary finite ring from the class of finite rings described by the construction and of characteristic p^4 with $p^2u_i = pv_j = pw_k = py_l = 0$. Then the group of units*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_4^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_2^e)^r \times (\mathbb{Z}_2^g)^r \times (\mathbb{Z}_8^{e+f})^r, & \text{if } p = 2 \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^3}^r \times (\mathbb{Z}_{p^2}^e)^r \times (\mathbb{Z}_p^f)^r \times (\mathbb{Z}_p^g)^r \times (\mathbb{Z}_p^h)^r, & \text{if } p \neq 2 \end{cases}$$

Proof. Since $R^* \cong \mathbb{Z}_{p^r-1} \times (1+Z(R))$, it suffices to determine the structure of $1+Z(R)$. Let $\varepsilon_1,\ldots,\varepsilon_r$ be elements of R_0 with $\varepsilon_1 = 1$ such that $\bar{\varepsilon}_1,\ldots,\bar{\varepsilon}_r$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbb{F}_p$. Then the generators with their respective orders are as indicated below:

Case(i): For $p=2$, $1 \le t \le r$, $1 \le i \le e$, $1 \le j \le f$, $1 \le k \le g$, the generators are $1+2\varepsilon_t$ of order 4; $1+6\varepsilon_t$ of oder 2, $1+2\varepsilon_t u_i$ of order 2; $1+\varepsilon_t w_k$ of order 2 and $1+\varepsilon_t u_i+\varepsilon_t v_j$ of order 8. The rest of the proof is similar to the proof of Proposition 3.7.

Case(ii): For $p \neq 2$, $1 \le t \le r$, $1 \le i \le e$, $1 \le j \le f$, $1 \le k \le g$, $1 \le l \le h$, the generators are $1+p\varepsilon_t$ of order p^3, $1+\varepsilon_t u_i$ of order p^2, $1+\varepsilon_t v_j$ of order p, $1+\varepsilon_t w_k$ of order p and $1+\varepsilon_t y_l$ of order p. The rest of the proof is similar to the proof of Proposition 3.7. □

Proposition 3.9. *Let R be a completely primary finite ring from the class of finite rings described by the construction and of characteristic p^4 with $p^3u_i = pv_j = pw_k = py_l = 0$. Then the group of units*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_4^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_8^e)^r \times (\mathbb{Z}_4^f)^r \times (\mathbb{Z}_2^g)^r, & \text{if } p = 2 \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^3}^r \times (\mathbb{Z}_{p^3}^e)^r \times (\mathbb{Z}_p^f)^r \times (\mathbb{Z}_p^g)^r \times (\mathbb{Z}_p^h)^r, & \text{if } p \neq 2 \end{cases}$$

Proof. Since $R^* \cong \mathbb{Z}_{p^r-1} \times (1+Z(R))$, it suffices to determine the structure of $1+Z(R)$. Let $\varepsilon_1, \ldots, \varepsilon_r$ be elements of R_0 with $\varepsilon_1 = 1$ such that $\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_r$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbb{F}_p$. Then the generators with their respective orders are as indicated below:

Case(i): For $p = 2$, $1 \le t \le r$, $1 \le i \le e$, $1 \le j \le f$, $1 \le k \le g$, the generators are $1+2\varepsilon_t$ of order 4; $1+6\varepsilon_t$ of order 2, $1+\varepsilon_t u_i$ of order 8; $1+\varepsilon_t v_j$ of order 4 and $1+\varepsilon_t w_k$ of order 2. The rest of the proof is similar to the proof of Proposition 3.7 .

Case(ii): For $p \ne 2$, $1 \le t \le r$, $1 \le i \le e$, $1 \le j \le f$, $1 \le k \le g$, $1 \le l \le h$, the generators are $1+p\varepsilon_t$ of order p^3, $1+\varepsilon_t u_i$ of order p^3, $1+\varepsilon_t v_j$ of order p, $1+\varepsilon_t w_k$ of order p and $1+\varepsilon_t y_l$ of order p. The rest of the proof is similar to the proof of Proposition 3.7. □

Proposition 3.10. *Let R be a completely primary finite ring from the class of finite rings described by the construction and of characteristic* p^4 *with* $p^4 u_i = pv_j = pw_k = py_l = 0$. *Then the group of units*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_4^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_{16}^e)^r \times (\mathbb{Z}_4^f)^r \times (\mathbb{Z}_2^g)^r, & if \;\; p=2 \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^3}^r \times (\mathbb{Z}_{p^4}^e)^r \times (\mathbb{Z}_p^f)^r \times (\mathbb{Z}_p^g)^r \times (\mathbb{Z}_p^h)^r, & if \;\; p \ne 2 \end{cases}$$

Proof. Since $R^* \cong \mathbb{Z}_{p^r-1} \times (1+Z(R))$, it suffices to determine the structure of $1+Z(R)$. Let $\varepsilon_1, \ldots, \varepsilon_r$ be elements of R_0 with $\varepsilon_1 = 1$ such that $\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_r$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbb{F}_p$. Then the generators with their respective orders are as indicated below:

Case(i): For $p = 2$, $1 \le t \le r$, $1 \le i \le e$, $1 \le j \le f$, $1 \le k \le g$, the generators are $1+2\varepsilon_t$ of order 4; $1+6\varepsilon_t$ of oder 2, $1+\varepsilon_t u_i$ of order 16; $1+\varepsilon_t v_j$ of order 4 and $1+\varepsilon_t w_k$ of order 2. The rest of the proof is similar to the proof of Proposition 3.7.

Case(ii): For $p \ne 2$, $1 \le t \le r$, $1 \le i \le e$, $1 \le j \le f$, $1 \le k \le g$, $1 \le l \le h$, the generators are $1+p\varepsilon_t$ of order p^3, $1+\varepsilon_t u_i$ of order p^4, $1+\varepsilon_t v_j$ of order p, $1+\varepsilon_t w_k$ of order p and $1+\varepsilon_t y_l$ of order p. The rest of the proof is similar to the proof of Proposition 3.7. □

Proposition 3.11. *Let R be a completely primary finite ring from the class of finite rings described by the construction and of characteristic* p^5 *with* $pu_i = pv_j = pw_k = py_l = 0$. *Then the group of units*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_8^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_8^e)^r \times (\mathbb{Z}_2^g)^r, & if \;\; p=2 \\ \mathbb{Z}_{3^r-1} \times \mathbb{Z}_{81}^r \times (\mathbb{Z}_9^e)^r \times (\mathbb{Z}_3^f)^r \times (\mathbb{Z}_3^h)^r, & if \;\; p=3 \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^4}^r \times (\mathbb{Z}_p^e)^r \times (\mathbb{Z}_p^f)^r \times (\mathbb{Z}_p^g)^r \times (\mathbb{Z}_p^h)^r, & if \;\; p>3 \end{cases}$$

Proof. Using the fact that $R^* \cong \mathbb{Z}_{p^r-1} \times (1+Z(R))$, it suffices to determine the structure of $1+Z(R)$. Let $\varepsilon_1, \ldots, \varepsilon_r$ be elements of R_0 with $\varepsilon_1 = 1$ such that $\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_r$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbb{F}_p$. We consider the three cases separately:

Case(i): $p = 2$. For every $t = 1, \ldots, r$, $(1+4\varepsilon_t)^8 = 1$, $(1+14\varepsilon_t)^2 = 1$, $(1+\varepsilon_t u_i)^8 = 1$, and $(1+\varepsilon_t w_k)^2 = 1$. For non-negative integers α_t, λ_t, φ_t and δ_t with $\alpha_t \le 8$, $\lambda_t \le 2$, $\varphi_t \le 8$ and $\delta_t \le 2$, it is clear that

$$\prod_{t=1}^{r} \{(1+4\varepsilon_t)^{\alpha_t}\} \cdot \prod_{t=1}^{r} \left\{(1+14\varepsilon_t)^{\delta_t}\right\} \cdot \prod_{i=1}^{e}\prod_{t=1}^{r} \{(1+\varepsilon_t u_i)^{\varphi_t}\} \cdot \prod_{k=1}^{g}\prod_{t=1}^{r} \left\{(1+\varepsilon_t w_k)^{\lambda_t}\right\} = \{1\}$$

This indicates that $\alpha_t = 8$, $\lambda_t = 2$, $\varphi_t = 8$ and $\delta_t = 2$ for all $t = 1, \ldots, r$.

Suppose

$A_t = \{(1+4\varepsilon_t)^{\alpha} : \alpha = 1,\ldots,8; \forall t = 1,\ldots,r\}$,
$B_t = \left\{(1+14\varepsilon_t)^{\delta} : \delta = 1,2; \forall t = 1,\ldots,r\right\}$,
$C_{ti} = \{(1+\varepsilon_t u_i)^{\varphi} : \varphi = 1,\ldots,8; \forall t = 1,\ldots,r\}$, and
$D_{tk} = \left\{(1+\varepsilon_t w_k)^{\lambda} : \lambda = 1,2; \forall t = 1,\ldots,r\right\}$,
then A_t, B_t, C_{ti} and D_{tk} are all cyclic subgroups of the group $1+Z(R)$ and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups $<1+4\varepsilon_t>$, $<1+14\varepsilon_t>$, $<1+\varepsilon_t u_i>$, and $<1+\varepsilon_t w_k>$ gives the identity group and that

$$\left|\prod_{t=1}^{r} <1+4\varepsilon_t>\right| \cdot \left|\prod_{t=1}^{r} <1+14\varepsilon_t>\right| \cdot \left|\prod_{i=1}^{e}\prod_{t=1}^{r} <1+\varepsilon_t u_i>\right| \cdot \left|\prod_{k=1}^{g}\prod_{t=1}^{r} <1+\varepsilon_t w_k>\right|$$

coincides with $|1+Z(R)|$, it follows that

$$\begin{aligned}1+Z(R) &= \prod_{t=1}^{r} <1+4\varepsilon_t> \times \prod_{t=1}^{r} <1+14\varepsilon_t> \times \prod_{i=1}^{e}\prod_{t=1}^{r} <1+\varepsilon_t u_i> \times \prod_{k=1}^{g}\prod_{t=1}^{r} <1+\varepsilon_t w_k> \\ &\cong \mathbb{Z}_8^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_8^e)^r \times (\mathbb{Z}_2^g)^r\end{aligned}$$

Case(ii): $p = 3$. For every $t = 1,\ldots,r$, $(1+3\varepsilon_t)^{81} = 1$, $(1+\varepsilon_t v_j)^3 = 1$, $(1+\varepsilon_t u_i)^9 = 1$, and $(1+\varepsilon_t y_l)^3 = 1$. For non-negative integers α_t, λ_t, φ_t and δ_t with $\alpha_t \le 81$, $\lambda_t \le 3$, $\varphi_t \le 9$ and $\delta_t \le 3$, it is clear that

$$\prod_{t=1}^{r}\{(1+3\varepsilon_t)^{\alpha_t}\} \cdot \prod_{i=1}^{e}\prod_{t=1}^{r}\{(1+\varepsilon_t u_i)^{\varphi_t}\} \cdot \prod_{j=1}^{f}\prod_{t=1}^{r}\left\{(1+\varepsilon_t v_j)^{\delta_t}\right\} \cdot \prod_{l=1}^{h}\prod_{t=1}^{r}\left\{(1+\varepsilon_t y_l)^{\lambda_t}\right\} = \{1\}$$

This indicates that $\alpha_t = 81$, $\lambda_t = 3$, $\varphi_t = 9$ and $\delta_t = 3$ for all $t = 1,\ldots,r$.
Suppose
$A_t = \{(1+3\varepsilon_t)^{\alpha} : \alpha = 1,\ldots,81; \forall t = 1,\ldots,r\}$,
$B_{ti} = \{(1+\varepsilon_t u_i)^{\varphi} : \varphi = 1,\ldots,9; \forall t = 1,\ldots,r\}$,
$C_{tj} = \left\{(1+\varepsilon_t v_j)^{\delta} : \delta = 1,2,3; \forall t = 1,\ldots,r\right\}$, and
$D_{tl} = \left\{(1+\varepsilon_t y_l)^{\lambda} : \lambda = 1,2,3; \forall t = 1,\ldots,r\right\}$,
then A_t, B_{ti}, C_{tj} and D_{tl} are all cyclic subgroups of the group $1+Z(R)$ and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups $<1+3\varepsilon_t>$, $<1+\varepsilon_t u_i>$, $<1+\varepsilon_t v_j>$, and $<1+\varepsilon_t y_l>$ gives the identity group and that

$$\left|\prod_{t=1}^{r} <1+3\varepsilon_t>\right| \cdot \left|\prod_{i=1}^{e}\prod_{t=1}^{r} <1+\varepsilon_t u_i>\right| \cdot \left|\prod_{j=1}^{f}\prod_{t=1}^{r} <1+\varepsilon_t v_j>\right| \cdot \left|\prod_{l=1}^{h}\prod_{t=1}^{r} <1+\varepsilon_t y_l>\right|$$

coincides with $|1+Z(R)|$, it follows that

$$\begin{aligned}1+Z(R) &= \prod_{t=1}^{r} <1+3\varepsilon_t> \times \prod_{i=1}^{e}\prod_{t=1}^{r} <1+\varepsilon_t u_i> \times \prod_{j=1}^{f}\prod_{t=1}^{r} <1+\varepsilon_t v_j> \times \prod_{l=1}^{h}\prod_{t=1}^{r} <1+\varepsilon_t y_l> \\ &\cong \mathbb{Z}_{81}^r \times (\mathbb{Z}_9^e)^r \times (\mathbb{Z}_3^f)^r \times (\mathbb{Z}_3^h)^r\end{aligned}$$

Case(iii): $p > 3$. For every $t = 1,\ldots,r$, $(1+p\varepsilon_t)^{p^4} = 1$, $(1+\varepsilon_t u_i)^p = 1$, $(1+\varepsilon_t v_j)^p = 1$, $(1+\varepsilon_t w_k)^p = 1$, and $(1+\varepsilon_t y_l)^p = 1$. For non-negative integers α_t, λ_t, φ_t, β_t, and δ_t with $\alpha_t \le p^4$, $\lambda_t \le p$, $\varphi_t \le p$, $\beta_t \le p$ and $\delta_t \le p$, it is clear that

$$\begin{aligned}&\prod_{t=1}^{r}\{(1+p\varepsilon_t)^{\alpha_t}\} \cdot \prod_{i=1}^{e}\prod_{t=1}^{r}\{(1+\varepsilon_t u_i)^{\varphi_t}\} \cdot \prod_{j=1}^{f}\prod_{t=1}^{r}\left\{(1+\varepsilon_t v_j)^{\delta_t}\right\} \cdot \prod_{k=1}^{g}\prod_{t=1}^{r}\left\{(1+\varepsilon_t w_k)^{\beta_t}\right\} \cdot \\ &\prod_{l=1}^{h}\prod_{t=1}^{r}\left\{(1+\varepsilon_t y_l)^{\lambda_t}\right\} = \{1\}\end{aligned}$$

This indicates that $\alpha_t = p^4,\ \lambda_t = p,\ \varphi_t = p,\ \delta_t = p$, and $\beta_t = p$, for all $t = 1, \ldots, r$.
Suppose
$A_t = \left\{(1 + p\varepsilon_t)^\alpha : \alpha = 1, \ldots, p^4; \forall t = 1, \ldots, r\right\}$,
$B_{ti} = \left\{(1 + \varepsilon_t u_i)^\varphi : \varphi = 1, \ldots, p; \forall t = 1, \ldots, r\right\}$,
$C_{tj} = \left\{(1 + \varepsilon_t v_j)^\delta : \delta = 1, \ldots, p; \forall t = 1, \ldots, r\right\}$,
$D_{tk} = \left\{(1 + \varepsilon_t w_k)^\beta : \beta = 1, \ldots, p; \forall t = 1, \ldots, r\right\}$, and
$E_{tl} = \left\{(1 + \varepsilon_t y_l)^\lambda : \lambda = 1, \ldots, p; \forall t = 1, \ldots, r\right\}$,
then $A_t,\ B_{ti}, C_{tj},\ D_{tk}$, and E_{tl}, are all cyclic subgroups of the group $1 + Z(R)$ and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups $< 1 + p\varepsilon_t >$, $< 1 + \varepsilon_t u_i >$, $< 1 + \varepsilon_t v_j >$, $< 1 + \varepsilon_t w_k >$, and $< 1 + \varepsilon_t y_l >$, gives the identity group and that

$$\left|\prod_{t=1}^{r} < 1 + p\varepsilon_t >\right| \cdot \left|\prod_{i=1}^{e}\prod_{t=1}^{r} < 1 + \varepsilon_t u_i >\right| \cdot \left|\prod_{j=1}^{f}\prod_{t=1}^{r} < 1 + \varepsilon_t v_j >\right| \cdot \left|\prod_{k=1}^{g}\prod_{t=1}^{r} < 1 + \varepsilon_t w_k >\right| \cdot \left|\prod_{l=1}^{h}\prod_{t=1}^{r} < 1 + \varepsilon_t y_l >\right|$$

coincides with $|1 + Z(R)|$, it follows that

$$\begin{aligned} 1 + Z(R) &= \prod_{t=1}^{r} < 1 + p\varepsilon_t > \times \prod_{i=1}^{e}\prod_{t=1}^{r} < 1 + \varepsilon_t u_i > \times \prod_{j=1}^{f}\prod_{t=1}^{r} < 1 + \varepsilon_t v_j > \\ &\times \prod_{k=1}^{g}\prod_{t=1}^{r} < 1 + \varepsilon_t w_k > \times \prod_{l=1}^{h}\prod_{t=1}^{r} < 1 + \varepsilon_t y_l > \\ &\cong \mathbb{Z}_{p^4}^r \times (\mathbb{Z}_p^e)^r \times (\mathbb{Z}_p^f)^r \times (\mathbb{Z}_p^g)^r \times (\mathbb{Z}_p^h)^r \end{aligned}$$

□

Proposition 3.12. *Let R be a completely primary finite ring from the class of finite rings described by the construction and of characteristic p^5 with $p^2 u_i = pv_j = pw_k = py_l = 0$. Then the group of units*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_8^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_2^e)^r \times (\mathbb{Z}_2^g)^r \times (\mathbb{Z}_8^{e+f})^r, & \text{if } p = 2 \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^4}^r \times (\mathbb{Z}_{p^2}^e)^r \times (\mathbb{Z}_p^f)^r \times (\mathbb{Z}_p^g)^r \times (\mathbb{Z}_p^h)^r, & \text{if } p \neq 2 \end{cases}$$

Proof. Since $R^* \cong \mathbb{Z}_{p^r-1} \times (1 + Z(R))$, it suffices to determine the structure of $1 + Z(R)$. Let $\varepsilon_1, \ldots, \varepsilon_r$ be elements of R_0 with $\varepsilon_1 = 1$ such that $\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_r$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbb{F}_p$. Then the generators with their respective orders are as indicated below:

Case(i): For $p = 2,\ 1 \leq t \leq r,\ 1 \leq i \leq e,\ 1 \leq j \leq f,\ 1 \leq k \leq g$, the generators are $1 + 4\varepsilon_t$ of order 8; $1 + 14\varepsilon_t$ of order 2, $1 + 2\varepsilon_t u_i$ of order 2; $1 + \varepsilon_t w_k$ of order 2 and $1 + \varepsilon_t u_i + \varepsilon_t v_j$ of order 8. The rest of the proof is similar to the proof of Proposition 3.11.

Case(ii): For $p \neq 2,\ 1 \leq t \leq r,\ 1 \leq i \leq e,\ 1 \leq j \leq f,\ 1 \leq k \leq g,\ 1 \leq l \leq h$, the generators are $1 + p\varepsilon_t$ of order p^4, $1 + \varepsilon_t u_i$ of order p^2, $1 + \varepsilon_t v_j$ of order p, $1 + \varepsilon_t w_k$ of order p and $1 + \varepsilon_t y_l$ of order p. The rest of the proof is similar to the proof of Proposition 3.11. □

Proposition 3.13. *Let R be a completely primary finite ring from the class of finite rings described by the construction and of characteristic p^5 with $p^3 u_i = pv_j = pw_k = py_l = 0$. Then the group of units*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_8^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_8^e)^r \times (\mathbb{Z}_4^f)^r \times (\mathbb{Z}_2^g)^r, & \text{if } p = 2 \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^4}^r \times (\mathbb{Z}_{p^3}^e)^r \times (\mathbb{Z}_p^f)^r \times (\mathbb{Z}_p^g)^r \times (\mathbb{Z}_p^h)^r, & \text{if } p \neq 2 \end{cases}$$

Proof. Since $R^* \cong \mathbb{Z}_{p^r-1} \times (1 + Z(R))$, it suffices to determine the structure of $1 + Z(R)$. Let $\varepsilon_1, \ldots, \varepsilon_r$ be elements of R_0 with $\varepsilon_1 = 1$ such that $\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_r$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbb{F}_p$. Then the generators with their respective orders are as indicated below:

Case(i): For $p = 2,\ 1 \le t \le r,\ 1 \le i \le e,\ 1 \le j \le f,\ 1 \le k \le g$, the generators are $1 + 4\varepsilon_t$ of order 8; $1 + 14\varepsilon_t$ of oder 2, $1 + \varepsilon_t u_i$ of order 8; $1 + \varepsilon_t v_j$ of order 4 and $1 + \varepsilon_t w_k$ of order 2. The rest of the proof is similar to the proof of Proposition 3.11.

Case(ii): For $p \ne 2,\ 1 \le t \le r,\ 1 \le i \le e,\ 1 \le j \le f,\ 1 \le k \le g,\ 1 \le l \le h$, the generators are $1 + p\varepsilon_t$ of order p^4, $1 + \varepsilon_t u_i$ of order p^3, $1 + \varepsilon_t v_j$ of order p, $1 + \varepsilon_t w_k$ of order p and $1 + \varepsilon_t y_l$ of order p. The rest of the proof is similar to the proof of Proposition 3.11. ☐

Proposition 3.14. *Let R be a completely primary finite ring from the class of finite rings described by the construction and of characteristic p^5 with $p^4 u_i = pv_j = pw_k = py_l = 0$. Then the group of units*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_8^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_{16}^e)^r \times (\mathbb{Z}_4^f)^r \times (\mathbb{Z}_2^g)^r, & \text{if } p = 2 \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^4}^r \times (\mathbb{Z}_{p^4}^e)^r \times (\mathbb{Z}_p^f)^r \times (\mathbb{Z}_p^g)^r \times (\mathbb{Z}_p^h)^r, & \text{if } p \ne 2 \end{cases}$$

Proof. Since $R^* \cong \mathbb{Z}_{p^r-1} \times (1 + Z(R))$, it suffices to determine the structure of $1 + Z(R)$. Let $\varepsilon_1, \ldots, \varepsilon_r$ be elements of R_0 with $\varepsilon_1 = 1$ such that $\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_r$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbb{F}_p$. Then the generators with their respective orders are as indicated below:

Case(i): For $p = 2,\ 1 \le t \le r,\ 1 \le i \le e,\ 1 \le j \le f,\ 1 \le k \le g$, the generators are $1 + 4\varepsilon_t$ of order 8; $1 + 14\varepsilon_t$ of order 2, $1 + \varepsilon_t u_i$ of order 16; $1 + \varepsilon_t v_j$ of order 4 and $1 + \varepsilon_t w_k$ of order 2. The rest of the proof is similar to the proof of Proposition 3.11.

Case(ii): For $p \ne 2,\ 1 \le t \le r,\ 1 \le i \le e,\ 1 \le j \le f,\ 1 \le k \le g,\ 1 \le l \le h$, the generators are $1 + p\varepsilon_t$ of order p^4, $1 + \varepsilon_t u_i$ of order p^4, $1 + \varepsilon_t v_j$ of order p, $1 + \varepsilon_t w_k$ of order p, and $1 + \varepsilon_t y_l$ of order p. The rest of the proof is similar to the proof of Proposition 3.11. ☐

4 Conclusion

This study has constructed a class of five radical zero commutative completely primary finite rings and classified its unit groups for some selected classes. This has been possible through isolation of the set of invertible elements from the set of zero divisors. classification of the group of units of other classes will be considered in subsequent work. For the characterization of zero divisors graphs for such rings, the publication is yet to appear. Since the classification of finite rings is still incomplete, future researchers may study rings whose subsets of zero divisors are of higher indices of nilpotence.

Acknowledgement

The first author thanks Egerton University for providing 100% tuition fee waiver for his Ph.D studies. The second author thanks University of Kabianga for granting him the opportunity to partake in the research.

Competing Interests

Authors have declared that no competing interests exist

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