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Properties of Generalized Fifth-Order Pell Numbers

Yüksel Soykan[1*]

[1] *Department of Mathematics, Art and Science Faculty, Zonguldak Bülent Ecevit University, 67100, Zonguldak, Turkey.*

Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

In this paper, we investigate the generalized fifth order Pell sequences and we deal with, in detail, three special cases which we call them as fifth order Pell, fifth order Pell-Lucas and modified fifth order Pell sequences.

Keywords: Pell numbers; fifth order Pell numbers; fifth order Pell-Lucas numbers.

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1 Introduction and Preliminaries

In this paper, we introduce the generalized fifth order Pell sequences and we investigate, in detail, three special cases which we call them fifth order Pell, fifth order Pell-Lucas and modified fifth order Pell sequences. First we recall the definition of a generalized Pentanacci sequence.

A generalized Pentanacci sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3, W_4; r_1, r_2, r_3, r_4, r_5)\}_{n\geq 0}$ is

**Corresponding author: E-mail: yuksel_soykan@hotmail.com;*

defined by the fifth-order recurrence relations

$$W_n = r_1 W_{n-1} + r_2 W_{n-2} + r_3 W_{n-3} + r_4 W_{n-4} + r_5 W_{n-5}, \qquad W_0 = a, W_1 = b, W_2 = c, W_3 = d, W_4 = e \tag{1.1}$$

where the initial values W_0, W_1, W_2, W_3, W_4 are arbitrary complex (or real) numbers and r_1, r_2, r_3, r_4, r_5 are real numbers. Pentanacci sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [1], [2], [3]. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r_4}{r_5} W_{-(n-1)} - \frac{r_3}{r_5} W_{-(n-2)} - \frac{r_2}{r_5} W_{-(n-3)} - \frac{r_1}{r_5} W_{-(n-4)} + \frac{1}{r_5} W_{-(n-5)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

It is well-known that the Pell sequence (A000129 in [4]) $\{P_n\}$ is defined recursively by the equation, for $n \geq 0$

$$P_{n+2} = 2P_{n+1} + P_n$$

in which $P_0 = 0$ and $P_1 = 1$. Then Pell sequence (second order Pell sequence) is

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, ...$$

Pell sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [5,6,7,8,9,10,11,12,13]. For higher order Pell sequences, see [14,15,16,17].

In this paper we consider the case $r_1 = 2$, $r_2 = r_3 = r_4 = r_5 = 1$ and in this case we write $V_n = W_n$. A generalized fifth order Pell sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2, V_3, V_4)\}_{n\geq 0}$ is defined by the fifth-order recurrence relations

$$V_n = 2V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5} \tag{1.2}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$, $V_3 = c_3$, $V_4 = c_4$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} - V_{-(n-2)} - V_{-(n-3)} - 2V_{-(n-4)} + V_{-(n-5)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.2) holds for all integer n.

As $\{V_n\}$ is a fifth order recurrence sequence (difference equation), it's characteristic equation is

$$x^5 - 2x^4 - x^3 - x^2 - x - 1 = 0. \tag{1.3}$$

The approximate value of the roots $\alpha, \beta, \gamma, \delta$ and λ of Equation (1.3) are given by

$$\begin{aligned}
\alpha &= 2.6083299 \\
\beta &= 0.282\,69438 - 0.794\,69421i \\
\gamma &= 0.282\,69438 + 0.794\,69421i \\
\delta &= -0.586\,85934 - 0.440\,99162i \\
\lambda &= -0.586\,85934 + 0.440\,99162i
\end{aligned}$$

Note that we have the following identities:

$$\begin{aligned}
\alpha + \beta + \gamma + \delta + \lambda &= 2, \\
\alpha\beta + \alpha\lambda + \alpha\gamma + \beta\lambda + \alpha\delta + \beta\gamma + \lambda\gamma + \beta\delta + \lambda\delta + \gamma\delta &= -1, \\
\alpha\beta\lambda + \alpha\beta\gamma + \alpha\lambda\gamma + \alpha\beta\delta + \alpha\lambda\delta + \beta\lambda\gamma + \alpha\gamma\delta + \beta\lambda\delta + \beta\gamma\delta + \lambda\gamma\delta &= 1, \\
\alpha\beta\lambda\gamma + \alpha\beta\lambda\delta + \alpha\beta\gamma\delta + \alpha\lambda\gamma\delta + \beta\lambda\gamma\delta &= -1 \\
\alpha\beta\gamma\delta\lambda &= 1.
\end{aligned}$$

The first few generalized fifth order Pell numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized fifth order Pell numbers

n	V_n	V_{-n}
0	V_0	V_0
1	V_1	$-V_0 - V_1 - V_2 - 2 \times V_3 + V_4$
2	V_2	$-V_4 + 3V_3 - V_2$
3	V_3	$-V_3 + 3V_2 - V_1$
4	V_4	$-V_2 + 3V_1 - V_0$
5	$2V_4 + V_3 + V_2 + V_1 + V_0$	$-V_4 + 2V_3 + V_2 + 4V_0$
6	$5V_4 + 3V_3 + 3V_2 + 3V_1 + 2V_0$	$4V_4 - 9V_3 - 2V_2 - 3V_1 - 4V_0$
7	$13V_4 + 8V_3 + 8V_2 + 7V_1 + 5V_0$	$-4V_4 + 12V_3 - 5V_2 + 2V_1 + V_0$
8	$34V_4 + 21V_3 + 20V_2 + 18V_1 + 13V_0$	$V_4 - 6V_3 + 11V_2 - 6V_1 + V_0$
9	$89V_4 + 54V_3 + 52V_2 + 47V_1 + 34V_0$	$V_4 - V_3 - 7V_2 + 10V_1 - 7V_0$
10	$232V_4 + 141V_3 + 136V_2 + 123V_1 + 89V_0$	$-7V_4 + 15V_3 + 6V_2 + 17V_0$

Now we define three special case of the sequence $\{V_n\}$. Fifth-order Pell sequence $\{P_n^{(5)}\}_{n\geq 0}$, fifth-order Pell-Lucas sequence $\{Q_n^{(5)}\}_{n\geq 0}$ and modified fifth-order Pell sequence $\{E_n^{(5)}\}_{n\geq 0}$ are defined, respectively, by the fifth-order recurrence relations

$$P_{n+5}^{(5)} = 2P_{n+4}^{(5)} + P_{n+3}^{(5)} + P_{n+2}^{(5)} + P_{n+1}^{(5)} + P_n^{(5)}, \qquad P_0^{(5)} = 0, P_1^{(5)} = 1, P_2^{(5)} = 2, P_3^{(5)} = 5, P_4^{(5)} = 13, \tag{1.4}$$

and

$$Q_{n+5}^{(5)} = 2Q_{n+4}^{(5)} + Q_{n+3}^{(5)} + Q_{n+2}^{(5)} + Q_{n+1}^{(5)} + Q_n^{(5)}, \qquad Q_0^{(5)} = 4, Q_1^{(5)} = 2, Q_2^{(5)} = 6, Q_3^{(5)} = 17, Q_4^{(5)} = 46, \tag{1.5}$$

and

$$E_{n+5}^{(5)} = 2E_{n+4}^{(5)} + E_{n+3}^{(5)} + E_{n+2}^{(5)} + E_{n+1}^{(5)} + E_n^{(5)}, \qquad E_0^{(5)} = 0, E_1^{(5)} = 1, E_2^{(5)} = 1, E_3^{(5)} = 3, E_4^{(5)} = 8. \tag{1.6}$$

The sequences $\{P_n^{(5)}\}_{n\geq 0}$, $\{Q_n^{(5)}\}_{n\geq 0}$ and $\{E_n^{(5)}\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n}^{(5)} = -P_{-(n-1)}^{(5)} - P_{-(n-2)}^{(5)} - P_{-(n-3)}^{(5)} - 2P_{-(n-4)}^{(5)} + P_{-(n-5)}^{(5)} \tag{1.7}$$

and

$$Q_{-n}^{(5)} = -Q_{-(n-1)}^{(5)} - Q_{-(n-2)}^{(5)} - Q_{-(n-3)}^{(5)} - 2Q_{-(n-4)}^{(5)} + Q_{-(n-5)}^{(5)} \tag{1.8}$$

and

$$E_{-n}^{(5)} = -E_{-(n-1)}^{(5)} - E_{-(n-2)}^{(5)} - E_{-(n-3)}^{(5)} - 2E_{-(n-4)}^{(5)} + E_{-(n-5)}^{(5)} \tag{1.9}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.7), (1.8) and (1.9) hold for all integer n.

In the rest of the paper, for easy writing, we drop the superscripts and write P_n, Q_n and E_n for $P_n^{(5)}, Q_n^{(5)}$ and $E_n^{(5)}$, respectively.

Note that P_n is the sequence A141448 in [4] and Q_n and E_n sequences are't in the database of http://oeis.org [4], yet.

Next, we present the first few values of the fifth-order Pell, fifth-order Pell-Lucas and modified fifth-order Pell numbers with positive and negative subscripts:

Table 2. The first few values of the special fifth-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
P_n	0	1	2	5	13	34	89	232	605	1578	4116	10736	28003	73041
P_{-n}	0	0	0	0	1	-1	0	0	-1	4	-4	1	1	-7
Q_n	5	2	6	17	46	122	315	821	2142	5588	14576	38018	99163	258650
Q_{-n}	5	-1	-1	-1	-5	14	-7	-1	3	-28	54	-34	1	38
E_n	0	1	1	3	8	21	55	143	373	973	2538	6620	17267	45038
E_{-n}	0	0	0	-1	2	-1	0	1	-5	8	-5	0	8	-24

2 Generating Functions

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} V_n x^n$ of the sequence V_n.

Lemma 2.1. *Suppose that $f_{V_n}(x) = \sum\limits_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized fifth-order Pell sequence $\{V_n\}_{n\geq 0}$. Then, $\sum\limits_{n=0}^{\infty} V_n x^n$ is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - V_0)x^2 + (V_3 - 2V_2 - V_1 - V_0)x^3 + (V_4 - 2V_3 - V_2 - V_1 - V_0)x^4}{(1 - 2x - x^2 - x^3 - x^4 - x^5)}. \tag{2.1}$$

Proof. Using the definition of generalized fifth-order Pell numbers and substracting $2x\sum_{n=0}^{\infty} V_n x^n$, $x^2\sum_{n=0}^{\infty} V_n x^n$ and $x^3\sum_{n=0}^{\infty} V_n x^n$ and $x^4\sum_{n=0}^{\infty} V_n x^n$ and $x^5\sum_{n=0}^{\infty} V_n x^n$ from $\sum_{n=0}^{\infty} V_n x^n$ we obtain

$$\begin{aligned}
& (1 - 2x - x^2 - x^3 - x^4 - x^5) f_{V_n}(x) \\
= & \sum_{n=0}^{\infty} V_n x^n - 2x\sum_{n=0}^{\infty} V_n x^n - x^2\sum_{n=0}^{\infty} V_n x^n - x^3\sum_{n=0}^{\infty} V_n x^n - x^4\sum_{n=0}^{\infty} V_n x^n - x^5\sum_{n=0}^{\infty} V_n x^n \\
= & \sum_{n=0}^{\infty} V_n x^n - 2\sum_{n=0}^{\infty} V_n x^{n+1} - \sum_{n=0}^{\infty} V_n x^{n+2} - \sum_{n=0}^{\infty} V_n x^{n+3} - \sum_{n=0}^{\infty} V_n x^{n+4} - \sum_{n=0}^{\infty} V_n x^{n+5} \\
= & \sum_{n=0}^{\infty} V_n x^n - 2\sum_{n=1}^{\infty} V_{n-1} x^n - \sum_{n=2}^{\infty} V_{n-2} x^n - \sum_{n=3}^{\infty} V_{n-3} x^n - \sum_{n=4}^{\infty} V_{n-4} x^n - \sum_{n=5}^{\infty} V_{n-5} x^n \\
= & (V_0 + V_1 x + V_2 x^2 + V_3 x^3 + V_4 x^4) - 2(V_0 x + V_1 x^2 + V_2 x^3 + V_3 x^4) - (V_0 x^2 + V_1 x^3 + V_2 x^4) \\
& -(V_0 x^3 + V_1 x^4) - V_0 x^4 + \sum_{n=5}^{\infty} (V_n - V_{n-1} - V_{n-2} - V_{n-3} - V_{n-4} - V_{n-5}) x^n \\
= & V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - V_0)x^2 + (V_3 - 2V_2 - V_1 - V_0)x^3 + (V_4 - 2V_3 - V_2 - V_1 - V_0)x^4.
\end{aligned}$$

Rearranging above equation, we get

$$f_{V_n}(x) = \frac{V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - V_0)x^2 + (V_3 - 2V_2 - V_1 - V_0)x^3 + (V_4 - 2V_3 - V_2 - V_1 - V_0)x^4}{(1 - 2x - x^2 - x^3 - x^4 - x^5)}.$$

The previous Lemma gives the following results as particular examples.

Corollary 2.2. *Generated functions of fifth-order Pell, Pell-Lucas and modified Pell numbers are*

$$\sum_{n=0}^{\infty} P_n x^n = \frac{x}{(1 - 2x - x^2 - x^3 - x^4 - x^5)},$$

and

$$\sum_{n=0}^{\infty} Q_n x^n = \frac{5 - 8x - 3x^2 - 2x^3 - x^4}{(1 - 2x - x^2 - x^3 - x^4 - x^5)},$$

and

$$\sum_{n=0}^{\infty} E_n x^n = \frac{x-x^2}{(1-2x-x^2-x^3-x^4-x^5)},$$

respectively.

3 Obtaning Binet Formula from Generating Function

We next find Binet formula of generalized fifth order Pell numbers $\{V_n\}$ by the use of generating function for V_n.

Theorem 3.1. *(Binet formula of generalized fifth order Pell numbers)*

$$\begin{aligned} V_n &= \frac{d_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{d_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} + \frac{d_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} && (3.1)\\ &+ \frac{d_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{d_5\delta^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)} && (3.2)\end{aligned}$$

where

$$\begin{aligned} d_1 &= V_0\alpha^4 + (V_1-2V_0)\alpha^3 + (V_2-2V_1-V_0)\alpha^2 + (V_3-2V_2-V_1-V_0)\alpha + (V_4-2V_3-V_2-V_1-V_0),\\ d_2 &= V_0\beta^4 + (V_1-2V_0)\beta^3 + (V_2-2V_1-V_0)\beta^2 + (V_3-2V_2-V_1-V_0)\beta + (V_4-2V_3-V_2-V_1-V_0),\\ d_3 &= V_0\gamma^4 + (V_1-2V_0)\gamma^3 + (V_2-2V_1-V_0)\gamma^2 + (V_3-2V_2-V_1-V_0)\gamma + (V_4-2V_3-V_2-V_1-V_0),\\ d_4 &= V_0\delta^4 + (V_1-2V_0)\delta^3 + (V_2-2V_1-V_0)\delta^2 + (V_3-2V_2-V_1-V_0)\delta + (V_4-2V_3-V_2-V_1-V_0),\\ d_5 &= V_0\lambda^4 + (V_1-2V_0)\lambda^3 + (V_2-2V_1-V_0)\lambda^2 + (V_3-2V_2-V_1-V_0)\lambda + (V_4-2V_3-V_2-V_1-V_0).\end{aligned}$$

Proof. Let

$$h(x) = 1-2x-x^2-x^3-x^4-x^5.$$

Then for some $\alpha,\beta,\gamma,\delta$ and λ we write

$$h(x) = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)(1-\lambda x)$$

i.e.,

$$1-2x-x^2-x^3-x^4-x^5 = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)(1-\lambda x) \qquad (3.3)$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$ ve $\frac{1}{\lambda}$ are the roots of $h(x)$. This gives $\alpha,\beta,\gamma,\delta$ and λ as the roots of

$$h(\frac{1}{x}) = 1-\frac{2}{x}-\frac{1}{x^2}-\frac{1}{x^3}-\frac{1}{x^4}-\frac{1}{x^5} = 0.$$

This implies $x^5-2x^4-x^3-x^2-x-1=0$. Now, by (2.1) and (3.3), it follows that

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0+(V_1-2V_0)x+(V_2-2V_1-V_0)x^2+(V_3-2V_2-V_1-V_0)x^3+(V_4-2V_3-V_2-V_1-V_0)x^4}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)(1-\lambda x)}.$$

Then we write

$$\begin{aligned} &\frac{V_0+(V_1-2V_0)x+(V_2-2V_1-V_0)x^2+(V_3-2V_2-V_1-V_0)x^3+(V_4-2V_3-V_2-V_1-V_0)x^4}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)(1-\lambda x)} && (3.4)\\ =\ &\frac{A_1}{(1-\alpha x)} + \frac{A_2}{(1-\beta x)} + \frac{A_3}{(1-\gamma x)} + \frac{A_4}{(1-\delta x)} + \frac{A_5}{(1-\lambda x)}.\end{aligned}$$

So

$$\begin{aligned} &V_0+(V_1-2V_0)x+(V_2-2V_1-V_0)x^2+(V_3-2V_2-V_1-V_0)x^3+(V_4-2V_3-V_2-V_1-V_0)x^4\\ =\ &A_1(1-\beta x)(1-\gamma x)(1-\delta x)(1-\lambda x) + A_2(1-\alpha x)(1-\gamma x)(1-\delta x)(1-\lambda x)\\ &+A_3(1-\alpha x)(1-\beta x)(1-\delta x)(1-\lambda x) + A_4(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\lambda x)\\ &+A_5(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x).\end{aligned}$$

If we consider $x = \frac{1}{\alpha}$, we get

$$\begin{aligned} & V_0 + (V_1 - 2V_0)\frac{1}{\alpha} + (V_2 - 2V_1 - V_0)\frac{1}{\alpha^2} + (V_3 - 2V_2 - V_1 - V_0)\frac{1}{\alpha^3} \\ +(V_4 - 2V_3 - V_2 - V_1 - V_0)\frac{1}{\alpha^4} &= A_1(1 - \frac{\beta}{\alpha})(1 - \frac{\gamma}{\alpha})(1 - \frac{\delta}{\alpha})(1 - \frac{\lambda}{\alpha}). \end{aligned}$$

This gives

$$\begin{aligned} A_1 &= \frac{\alpha^4(V_0 + (V_1 - 2V_0)\frac{1}{\alpha} + (V_2 - 2V_1 - V_0)\frac{1}{\alpha^2} + (V_3 - 2V_2 - V_1 - V_0)\frac{1}{\alpha^3} + (V_4 - 2V_3 - V_2 - V_1 - V_0)\frac{1}{\alpha^4})}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)(\alpha - \lambda)} \\ &= \frac{V_0\alpha^4 + (V_1 - 2V_0)\alpha^3 + (V_2 - 2V_1 - V_0)\alpha^2 + (V_3 - 2V_2 - V_1 - V_0)\alpha + (V_4 - 2V_3 - V_2 - V_1 - V_0)}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)(\alpha - \lambda)} \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} A_2 &= \frac{V_0\beta^4 + (V_1 - 2V_0)\beta^3 + (V_2 - 2V_1 - V_0)\beta^2 + (V_3 - 2V_2 - V_1 - V_0)\beta + (V_4 - 2V_3 - V_2 - V_1 - V_0)}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)(\beta - \lambda)}, \\ A_3 &= \frac{V_0\gamma^4 + (V_1 - 2V_0)\gamma^3 + (V_2 - 2V_1 - V_0)\gamma^2 + (V_3 - 2V_2 - V_1 - V_0)\gamma + (V_4 - 2V_3 - V_2 - V_1 - V_0)}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)(\gamma - \lambda)} \\ A_4 &= \frac{V_0\delta^4 + (V_1 - 2V_0)\delta^3 + (V_2 - 2V_1 - V_0)\delta^2 + (V_3 - 2V_2 - V_1 - V_0)\delta + (V_4 - 2V_3 - V_2 - V_1 - V_0)}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)(\delta - \lambda)} \\ A_5 &= \frac{V_0\lambda^4 + (V_1 - 2V_0)\lambda^3 + (V_2 - 2V_1 - V_0)\lambda^2 + (V_3 - 2V_2 - V_1 - V_0)\lambda + (V_4 - 2V_3 - V_2 - V_1 - V_0)}{(\lambda - \alpha)(\lambda - \beta)(\lambda - \gamma)(\lambda - \delta)}. \end{aligned}$$

Thus (3.4) can be written as

$$\sum_{n=0}^{\infty} V_n x^n = A_1(1 - \alpha x)^{-1} + A_2(1 - \beta x)^{-1} + A_3(1 - \gamma x)^{-1} + A_4(1 - \delta x)^{-1} + A_5(1 - \lambda x)^{-1}.$$

This gives

$$\begin{aligned} \sum_{n=0}^{\infty} V_n x^n &= A_1 \sum_{n=0}^{\infty} \alpha^n x^n + A_2 \sum_{n=0}^{\infty} \beta^n x^n + A_3 \sum_{n=0}^{\infty} \gamma^n x^n + A_4 \sum_{n=0}^{\infty} \delta^n x^n + A_5 \sum_{n=0}^{\infty} \lambda^n x^n \\ &= \sum_{n=0}^{\infty} (A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4\delta^n + A_5\lambda^n) x^n. \end{aligned}$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$V_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4\delta^n + A_5\lambda^n$$

and then we get (3.1).

Next, using Theorem 3.1, we present the Binet formulas of fifth-order Pell, Pell-Lucas and modified Pell sequences.

Corollary 3.2. *Binet formulas of fifth-order Pell, Pell-Lucas and modified Pell sequences are*

$$\begin{aligned} P_n &= \frac{\alpha^{n+3}}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)(\alpha - \lambda)} + \frac{\beta^{n+3}}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)(\beta - \lambda)} + \frac{\gamma^{n+3}}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)(\gamma - \lambda)} \\ &\quad + \frac{\delta^{n+3}}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)(\delta - \lambda)} + \frac{\lambda^{n+3}}{(\lambda - \alpha)(\lambda - \beta)(\lambda - \gamma)(\lambda - \delta)}, \end{aligned}$$

and

$$Q_n = \alpha^n + \beta^n + \gamma^n + \delta^n + \lambda^n,$$

and

$$\begin{aligned} E_n &= \frac{(\alpha - 1)\alpha^{n+2}}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)(\alpha - \lambda)} + \frac{(\beta - 1)\beta^{n+2}}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)(\beta - \lambda)} + \frac{(\gamma - 1)\gamma^{n+2}}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)(\gamma - \lambda)} \\ &\quad + \frac{(\delta - 1)\delta^{n+2}}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)(\delta - \lambda)} + \frac{(\lambda - 1)\lambda^{n+2}}{(\lambda - \alpha)(\lambda - \beta)(\lambda - \gamma)(\lambda - \delta)}, \end{aligned}$$

respectively.

Using the Binet formula of fifth-order Pell sequence, it can be shown that

$$\lim_{n\to\infty}\frac{P_{n+1}}{P_n}=\alpha=2.6083299.$$

Note that Binet formula of generalized fifth order Pell numbers can be represented as

$$\begin{aligned} V_n &= \frac{\alpha d_1\alpha^n}{2\alpha^4+2\alpha^3+3\alpha^2+4\alpha+5}+\frac{\beta d_2\beta^n}{2\beta^4+2\beta^3+3\beta^2+4\beta+5}+\frac{\gamma d_3\gamma^n}{2\gamma^4+2\gamma^3+3\gamma^2+4\gamma+5} \\ &\quad +\frac{\delta d_4\delta^n}{2\delta^4+2\delta^3+3\delta^2+4\delta+5}+\frac{\lambda d_5\lambda^n}{2\lambda^4+2\lambda^3+3\lambda^2+4\lambda+5} \end{aligned} \tag{3.5}$$

which can be derived from a result ((4.20) in page 25) of Hanusa [18]. When we compare (3.1) and (3.5), we see the following identities:

$$\begin{aligned} \frac{1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} &= \frac{\alpha}{2\alpha^4+2\alpha^3+3\alpha^2+4\alpha+5} \\ \frac{1}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} &= \frac{\beta}{2\beta^4+2\beta^3+3\beta^2+4\beta+5} \\ \frac{1}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} &= \frac{\gamma}{2\gamma^4+2\gamma^3+3\gamma^2+4\gamma+5} \\ \frac{1}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} &= \frac{\delta}{2\delta^4+2\delta^3+3\delta^2+4\delta+5} \\ \frac{1}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)} &= \frac{\lambda}{2\lambda^4+2\lambda^3+3\lambda^2+4\lambda+5} \end{aligned}$$

Using the above identities, we can give the Binet formulas of fifth-order Pell, Pell-Lucas and modified Pell sequences in the following form: Binet formulas of fifth-order Pell, Pell-Lucas and modified Pell sequences are

$$\begin{aligned} P_n &= \frac{\alpha^{n+4}}{2\alpha^4+2\alpha^3+3\alpha^2+4\alpha+5}+\frac{\beta^{n+4}}{2\beta^4+2\beta^3+3\beta^2+4\beta+5}+\frac{\gamma^{n+4}}{2\gamma^4+2\gamma^3+3\gamma^2+4\gamma+5} \\ &\quad +\frac{\delta^{n+4}}{2\delta^4+2\delta^3+3\delta^2+4\delta+5}+\frac{\lambda^{n+4}}{2\lambda^4+2\lambda^3+3\lambda^2+4\lambda+5}, \end{aligned}$$

and

$$Q_n=\alpha^n+\beta^n+\gamma^n+\delta^n+\lambda^n,$$

and

$$\begin{aligned} E_n &= \frac{(\alpha-1)\alpha^{n+3}}{2\alpha^4+2\alpha^3+3\alpha^2+4\alpha+5}+\frac{(\beta-1)\beta^{n+3}}{2\beta^4+2\beta^3+3\beta^2+4\beta+5}+\frac{(\gamma-1)\gamma^{n+3}}{2\gamma^4+2\gamma^3+3\gamma^2+4\gamma+5} \\ &\quad +\frac{(\delta-1)\delta^{n+3}}{2\delta^4+2\delta^3+3\delta^2+4\delta+5}+\frac{(\lambda-1)\lambda^{n+3}}{2\lambda^4+2\lambda^3+3\lambda^2+4\lambda+5}. \end{aligned}$$

respectively.

We can also find Binet formulas by using matrix method which is given in [15]. Take $k=i=5$ in Corollary 3.1 in [15]. Let

$$\Lambda = \begin{pmatrix} \alpha^4 & \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^4 & \beta^3 & \beta^2 & \beta & 1 \\ \gamma^4 & \gamma^3 & \gamma^2 & \gamma & 1 \\ \delta^4 & \delta^3 & \delta^2 & \delta & 1 \\ \lambda^4 & \lambda^3 & \lambda^2 & \lambda & 1 \end{pmatrix}, \Lambda_1 = \begin{pmatrix} \alpha^{n-1} & \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^{n-1} & \beta^3 & \beta^2 & \beta & 1 \\ \gamma^{n-1} & \gamma^3 & \gamma^2 & \gamma & 1 \\ \delta^{n-1} & \delta^3 & \delta^2 & \delta & 1 \\ \lambda^{n-1} & \lambda^3 & \lambda^2 & \lambda & 1 \end{pmatrix}, \Lambda_2 = \begin{pmatrix} \alpha^4 & \alpha^{n-1} & \alpha^2 & \alpha & 1 \\ \beta^4 & \beta^{n-1} & \beta^2 & \beta & 1 \\ \gamma^4 & \gamma^{n-1} & \gamma^2 & \gamma & 1 \\ \delta^4 & \delta^{n-1} & \delta^2 & \delta & 1 \\ \lambda^4 & \lambda^{n-1} & \lambda^2 & \lambda & 1 \end{pmatrix}$$

$$\Lambda_3 = \begin{pmatrix} \alpha^4 & \alpha^3 & \alpha^{n-1} & \alpha & 1 \\ \beta^4 & \beta^3 & \beta^{n-1} & \beta & 1 \\ \gamma^4 & \gamma^3 & \gamma^{n-1} & \gamma & 1 \\ \delta^4 & \delta^3 & \delta^{n-1} & \delta & 1 \\ \lambda^4 & \lambda^3 & \lambda^{n-1} & \lambda & 1 \end{pmatrix}, \Lambda_4 = \begin{pmatrix} \alpha^4 & \alpha^3 & \alpha^2 & \alpha^{n-1} & 1 \\ \beta^4 & \beta^3 & \beta^2 & \beta^{n-1} & 1 \\ \gamma^4 & \gamma^3 & \gamma^2 & \gamma^{n-1} & 1 \\ \delta^4 & \delta^3 & \delta^2 & \delta^{n-1} & 1 \\ \lambda^4 & \lambda^3 & \lambda^2 & \lambda^{n-1} & 1 \end{pmatrix}, \Lambda_5 = \begin{pmatrix} \alpha^4 & \alpha^3 & \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^4 & \beta^3 & \beta^2 & \beta & \beta^{n-1} \\ \gamma^4 & \gamma^3 & \gamma^2 & \gamma & \gamma^{n-1} \\ \delta^4 & \delta^3 & \delta^2 & \delta & \delta^{n-1} \\ \lambda^4 & \lambda^3 & \lambda^2 & \lambda & \lambda^{n-1} \end{pmatrix}.$$

Then the Binet formula for fifth-order Pell numbers is

$$\begin{aligned} P_n &= \frac{1}{\det(\Lambda)}\sum_{j=1}^{5} P_{6-j}\det(\Lambda_j) = \frac{1}{\Lambda}(P_5\det(\Lambda_1) + P_4\det(\Lambda_2) + P_3\det(\Lambda_3) + P_2\det(\Lambda_4) + P_1\det(\Lambda_5)) \\ &= \frac{1}{\det(\Lambda)}(34\det(\Lambda_1) + 13\det(\Lambda_2) + 5\det(\Lambda_3) + 2\det(\Lambda_4) + \det(\Lambda_5)) \end{aligned}$$

Similarly, we obtain the Binet formula for fifth-order Pell-Lucas and modified fifth-order Pell numbers as

$$\begin{aligned} Q_n &= \frac{1}{\Lambda}(Q_5\det(\Lambda_1) + Q_4\det(\Lambda_2) + Q_3\det(\Lambda_3) + Q_2\det(\Lambda_4) + Q_1\det(\Lambda_5)) \\ &= \frac{1}{\Lambda}(122\det(\Lambda_1) + 46\det(\Lambda_2) + 17\det(\Lambda_3) + 6\det(\Lambda_4) + 2\det(\Lambda_5)) \\ &= \alpha^n + \beta^n + \gamma^n + \delta^n + \lambda^n \end{aligned}$$

and

$$\begin{aligned} E_n &= \frac{1}{\Lambda}(E_5\det(\Lambda_1) + E_4\det(\Lambda_2) + E_3\det(\Lambda_3) + E_2\det(\Lambda_4) + E_1\det(\Lambda_5)) \\ &= \frac{1}{\Lambda}(21\det(\Lambda_1) + 8\det(\Lambda_2) + 3\det(\Lambda_3) + \det(\Lambda_4) + \det(\Lambda_5)) \end{aligned}$$

respectively.

4 Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following Theorem gives generalization of this result to the generalized Pentanacci sequence $\{W_n\}$.

Theorem 4.1 (Simson Formula of Generalized Pentanacci Numbers). *For all integers n we have*

$$\begin{vmatrix} W_{n+4} & W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+3} & W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} & W_{n-3} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} & W_{n-4} \end{vmatrix} = (r_5)^n \begin{vmatrix} W_4 & W_3 & W_2 & W_1 & W_0 \\ W_3 & W_2 & W_1 & W_0 & W_{-1} \\ W_2 & W_1 & W_0 & W_{-1} & W_{-2} \\ W_1 & W_0 & W_{-1} & W_{-2} & W_{-3} \\ W_0 & W_{-1} & W_{-2} & W_{-3} & W_{-4} \end{vmatrix}. \quad (4.1)$$

Proof. (4.1) is given in Soykan [19, Theorem 5].

A special case of the above theorem is the following Theorem which gives Simson formula of the generalized fifth-order Pell sequence $\{V_n\}$.

Theorem 4.2 (Simson Formula of Generalized Fifth-Order Pell Numbers). *For all integers n we have*

$$\begin{vmatrix} V_{n+4} & V_{n+3} & V_{n+2} & V_{n+1} & V_n \\ V_{n+3} & V_{n+2} & V_{n+1} & V_n & V_{n-1} \\ V_{n+2} & V_{n+1} & V_n & V_{n-1} & V_{n-2} \\ V_{n+1} & V_n & V_{n-1} & V_{n-2} & V_{n-3} \\ V_n & V_{n-1} & V_{n-2} & V_{n-3} & V_{n-4} \end{vmatrix} = \begin{vmatrix} V_4 & V_3 & V_2 & V_1 & V_0 \\ V_3 & V_2 & V_1 & V_0 & V_{-1} \\ V_2 & V_1 & V_0 & V_{-1} & V_{-2} \\ V_1 & V_0 & V_{-1} & V_{-2} & V_{-3} \\ V_0 & V_{-1} & V_{-2} & V_{-3} & V_{-4} \end{vmatrix}. \quad (4.2)$$

The previous Theorem gives the following results as particular examples.

Corollary 4.3. *Simson formula of fifth-order Pell, Pell-Lucas and modified Pell numbers are given as*

$$\begin{vmatrix} P_{n+4} & P_{n+3} & P_{n+2} & P_{n+1} & P_n \\ P_{n+3} & P_{n+2} & P_{n+1} & P_n & P_{n-1} \\ P_{n+2} & P_{n+1} & P_n & P_{n-1} & P_{n-2} \\ P_{n+1} & P_n & P_{n-1} & P_{n-2} & P_{n-3} \\ P_n & P_{n-1} & P_{n-2} & P_{n-3} & P_{n-4} \end{vmatrix} = 1. \tag{4.3}$$

and

$$\begin{vmatrix} Q_{n+4} & Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_n \\ Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_n & Q_{n-1} \\ Q_{n+2} & Q_{n+1} & Q_n & Q_{n-1} & Q_{n-2} \\ Q_{n+1} & Q_n & Q_{n-1} & Q_{n-2} & Q_{n-3} \\ Q_n & Q_{n-1} & Q_{n-2} & Q_{n-3} & Q_{n-4} \end{vmatrix} = 31409, \tag{4.4}$$

and

$$\begin{vmatrix} E_{n+4} & E_{n+3} & E_{n+2} & E_{n+1} & E_n \\ E_{n+3} & E_{n+2} & E_{n+1} & E_n & E_{n-1} \\ E_{n+2} & E_{n+1} & E_n & E_{n-1} & E_{n-2} \\ E_{n+1} & E_n & E_{n-1} & E_{n-2} & E_{n-3} \\ E_n & E_{n-1} & E_{n-2} & E_{n-3} & E_{n-4} \end{vmatrix} = 5, \tag{4.5}$$

respectively.

5 Some Identities

In this section, we obtain some identities of fifth order Pell, fifth order Pell-Lucas and modified fifth order Pell numbers. First, we can give a few basic relations between $\{P_n\}$ and $\{Q_n\}$.

Lemma 5.1. *The following equalities are true:*

$$\begin{aligned} Q_n &= 14P_{n+6} - 33P_{n+5} - 5P_{n+4} - 8P_{n+3} - 7P_{n+2} \qquad (5.1) \\ Q_n &= -5P_{n+5} + 9P_{n+4} + 6P_{n+3} + 7P_{n+2} + 14P_{n+1} \\ Q_n &= -P_{n+4} + P_{n+3} + 2P_{n+2} + 9P_{n+1} - 5P_n \\ Q_n &= -P_{n+3} + P_{n+2} + 8P_{n+1} - 6P_n - P_{n-1} \\ Q_n &= -P_{n+2} + 7P_{n+1} - 7P_n - 2P_{n-1} - P_{n-2} \end{aligned}$$

and

$$\begin{aligned} 4487P_n &= 121Q_{n+6} - 718Q_{n+5} + 1056Q_{n+4} + 37Q_{n+3} + 46Q_{n+2}, \\ 4487P_n &= -476Q_{n+5} + 1177Q_{n+4} + 158Q_{n+3} + 167Q_{n+2} + 121Q_{n+1}, \\ 4487P_n &= 225Q_{n+4} - 318Q_{n+3} - 309Q_{n+2} - 355Q_{n+1} - 476Q_n, \\ 4487P_n &= 132Q_{n+3} - 84Q_{n+2} - 130Q_{n+1} - 251Q_n + 225Q_{n-1}, \\ 4487P_n &= 180Q_{n+2} + 2Q_{n+1} - 119Q_n + 357Q_{n-1} + 132Q_{n-2}. \end{aligned}$$

Proof. Note that all the identities hold for all integers n. We prove (5.1). To show (5.1), writing

$$Q_n = a \times P_{n+6} + b \times P_{n+5} + c \times P_{n+4} + d \times P_{n+3} + e \times P_{n+2}$$

and solving the system of equations

$$\begin{aligned}
Q_0 &= a \times P_6 + b \times P_5 + c \times P_4 + d \times P_3 + e \times P_2 \\
Q_1 &= a \times P_7 + b \times P_6 + c \times P_5 + d \times P_4 + e \times P_3 \\
Q_2 &= a \times P_8 + b \times P_7 + c \times P_6 + d \times P_5 + e \times P_4 \\
Q_3 &= a \times P_9 + b \times P_8 + c \times P_7 + d \times P_6 + e \times P_5 \\
Q_4 &= a \times P_{10} + b \times P_9 + c \times P_8 + d \times P_7 + e \times P_6
\end{aligned}$$

we find that $a = 14, b = -33, c = -5, d = -8, e = -7$. The other equalities can be proved similarly.

Note that all the identities in the above Lemma can be proved by induction as well.

Secondly, we present a few basic relations between $\{P_n\}$ and $\{E_n\}$.

Lemma 5.2. *The following equalities are true:*

$$\begin{aligned}
E_n &= -P_{n+6} + 4P_{n+5} - 4P_{n+4} + P_{n+3} \\
E_n &= 2P_{n+5} - 5P_{n+4} - P_{n+2} - P_{n+1} \\
E_n &= -P_{n+4} + 2P_{n+3} + P_{n+2} + P_{n+1} + 2P_n \\
E_n &= P_n - P_{n-1}
\end{aligned}$$

and

$$\begin{aligned}
5P_n &= -4E_{n+6} + 9E_{n+5} + 3E_{n+4} + 2E_{n+3} + E_{n+2} \\
5P_n &= E_{n+5} - E_{n+4} - 2E_{n+3} - 3E_{n+2} - 4E_{n+1} \\
5P_n &= E_{n+4} - E_{n+3} - 2E_{n+2} - 3E_{n+1} + E_n \\
5P_n &= E_{n+3} - E_{n+2} - 2E_{n+1} + 2E_n + E_{n-1}.
\end{aligned}$$

Thirdly, we give a few basic relations between $\{Q_n\}$ and $\{E_n\}$.

Lemma 5.3. *The following equalities are true:*

$$\begin{aligned}
5Q_n &= 31E_{n+6} - 56E_{n+5} - 42E_{n+4} - 43E_{n+3} - 39E_{n+2} \\
5Q_n &= 6E_{n+5} - 11E_{n+4} - 12E_{n+3} - 8E_{n+2} + 31E_{n+1} \\
5Q_n &= E_{n+4} - 6E_{n+3} - 2E_{n+2} + 37E_{n+1} + 6E_n \\
5Q_n &= -4E_{n+3} - E_{n+2} + 38E_{n+1} + 7E_n + E_{n-1} \\
5Q_n &= -9E_{n+2} + 34E_{n+1} + 3E_n - 3E_{n-1} - 4E_{n-2}
\end{aligned}$$

and

$$\begin{aligned}
4487E_n &= 75Q_{n+6} - 747Q_{n+5} + 1820Q_{n+4} - 973Q_{n+3} + 55Q_{n+2} \\
4487E_n &= -597Q_{n+5} + 1895Q_{n+4} - 898Q_{n+3} + 130Q_{n+2} + 75Q_{n+1} \\
4487E_n &= 701Q_{n+4} - 1495Q_{n+3} - 467Q_{n+2} - 522Q_{n+1} - 597Q_n \\
4487E_n &= -93Q_{n+3} + 234Q_{n+2} + 179Q_{n+1} + 104Q_n + 701Q_{n-1} \\
4487E_n &= 48Q_{n+2} + 86Q_{n+1} + 11Q_n + 608Q_{n-1} - 93Q_{n-2}.
\end{aligned}$$

We now present a few special identities for the modified fifth order Pell sequence $\{E_n\}$.

Theorem 5.4. *(Catalan's identity) For all natural numbers n and m, the following identity holds*

$$E_{n+m}E_{n-m} - E_n^2 = (P_{n+m} - P_{n+m-1})(P_{n-m} - P_{n-m-1}) - (P_n - P_{n-1})^2$$

Proof. We use the identity

$$E_n = P_n - P_{n-1}.$$

Note that for $m = 1$ in Catalan's identity, we get the Cassini identity for the modified fifth order Pell sequence.

Corollary 5.5. *(Cassini's identity) For all natural numbers n and m, the following identity holds*

$$E_{n+1}E_{n-1} - E_n^2 = (P_{n+1} - P_n)(P_{n-1} - P_{n-2}) - (P_n - P_{n-1})^2.$$

The d'Ocagne's, Gelin-Cesàro's and Melham' identities can also be obtained by using $E_n = P_n - P_{n-1}$.The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of modified fifth order Pell sequence $\{E_n\}$.

Theorem 5.6. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$E_{m+1}E_n - E_mE_{n+1} = (P_{m+1} - P_m)(P_n - P_{n-1}) - (P_m - P_{m-1})(P_{n+1} - P_n).$$

(b) *(Gelin-Cesàro's identity)*

$$E_{n+2}E_{n+1}E_{n-1}E_{n-2} - E_n^4 = (P_{n+2} - P_{n+1})(P_{n+1} - P_n)(P_{n-1} - P_{n-2})(P_{n-2} - P_{n-3}) - (P_n - P_{n-1})^4$$

(c) *(Melham's identity)*

$$E_{n+1}E_{n+2}E_{n+6} - E_{n+3}^3 = (P_{n+1} - P_n)(P_{n+2} - P_{n+1})(P_{n+6} - P_{n+5}) - (P_{n+3} - P_{n+2})^3$$

Proof. Use the identity $E_n = P_n - P_{n-1}$.

6 Linear Sums

The following Theorem presents summing formulas of generalized fifth order Pell numbers.

Theorem 6.1. *For $n \geq 0$ we have the following formulas:*

(a) *(Sum of the generalized fifth order Pell numbers)*

$$\sum_{k=0}^{n} V_k = \frac{1}{5}(V_{n+5} - V_{n+4} - 2V_{n+3} - 3V_{n+2} - 4V_{n+1} - V_4 + V_3 + 2V_2 + 3V_1 + 4V_0)$$

(b)

$$\sum_{k=0}^{n} V_{2k} = \frac{1}{15}(-V_{2n+2} + 6V_{2n+1} + 7V_{2n} + 3V_{2n-1} + 4V_{2n-2} + V_4 - 6V_3 + 8V_2 - 3V_1 + 11V_0)$$

(c)

$$\sum_{k=0}^{n} V_{2k+1} = \frac{1}{15}(4V_{2n+2} + 6V_{2n+1} + 2V_{2n} + 3V_{2n-1} - V_{2n-2} - 4V_4 + 9V_3 - 2V_2 + 12V_1 + V_0).$$

Proof.

(a) Using the recurrence relation

$$V_n = 2V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5}$$

i.e.

$$V_{n-5} = V_n - 2V_{n-1} - V_{n-2} - V_{n-3} - V_{n-4}$$

we obtain

$$\begin{array}{rcl}
V_0 & = & V_5 - 2V_4 - V_3 - V_2 - V_1 \\
V_1 & = & V_6 - 2V_5 - V_4 - V_3 - V_2 \\
V_2 & = & V_7 - 2V_6 - V_5 - V_4 - V_3 \\
 & & \vdots \\
V_{n-5} & = & V_n - 2V_{n-1} - V_{n-2} - V_{n-3} - V_{n-4} \\
V_{n-4} & = & V_{n+1} - 2V_n - V_{n-1} - V_{n-2} - V_{n-3} \\
V_{n-3} & = & V_{n+2} - 2V_{n+1} - V_n - V_{n-1} - V_{n-2} \\
V_{n-2} & = & V_{n+3} - 2V_{n+2} - V_{n+1} - V_n - V_{n-1} \\
V_{n-1} & = & V_{n+4} - 2V_{n+3} - V_{n+2} - V_{n+1} - V_n \\
V_n & = & V_{n+5} - 2V_{n+4} - V_{n+3} - V_{n+2} - V_{n+1}.
\end{array}$$

If we add the above equations by side by, we get

$$\begin{array}{rcl}
\sum_{k=0}^{n} V_k & = & (V_{n+5} + V_{n+4} + V_{n+3} + V_{n+2} + V_{n+1} - V_4 - V_3 - V_2 - V_1 - V_0 + \sum_{k=0}^{n} V_k) \\
 & & -2(V_{n+4} + V_{n+3} + V_{n+2} + V_{n+1} - V_3 - V_2 - V_1 - V_0 + \sum_{k=0}^{n} V_k) \\
 & & -(V_{n+3} + V_{n+2} + V_{n+1} - V_2 - V_1 - V_0 + \sum_{k=0}^{n} V_k) \\
 & & -(V_{n+2} + V_{n+1} - V_1 - V_0 + \sum_{k=0}^{n} V_k) \\
 & & -(V_{n+1} - V_0 + \sum_{k=0}^{n} V_k)
\end{array}$$

Then, solving the above equality we obtain

$$\sum_{k=0}^{n} V_k = \frac{1}{5}(V_{n+5} - V_{n+4} - 2V_{n+3} - 3V_{n+2} - 4V_{n+1} - V_4 + V_3 + 2V_2 + 3V_1 + 4V_0).$$

(b) and (c) Using the recurrence relation

$$V_n = 2V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5}$$

i.e.

$$2V_{n-1} = V_n - V_{n-2} - V_{n-3} - V_{n-4} - V_{n-5}$$

we obtain

$$\begin{aligned}
2V_3 &= V_4 - V_2 - V_1 - V_0 - V_{-1} \\
2V_5 &= V_6 - V_4 - V_3 - V_2 - V_1 \\
2V_7 &= V_8 - V_6 - V_5 - V_4 - V_3 \\
&\vdots \\
2V_{2n-1} &= V_{2n} - V_{2n-2} - V_{2n-3} - V_{2n-4} - V_{2n-5} \\
2V_{2n+1} &= V_{2n+2} - V_{2n} - V_{2n-1} - V_{2n-2} - V_{2n-3} \\
2V_{2n+3} &= V_{2n+4} - V_{2n+2} - V_{2n+1} - V_{2n} - V_{2n-1} \\
2V_{2n+5} &= V_{2n+6} - V_{2n+4} - V_{2n+3} - V_{2n+2} - V_{2n+1}.
\end{aligned}$$

Now, if we add the above equations by side by, we get

$$\begin{aligned}
2(-V_1 + \sum_{k=0}^{n} V_{2k+1}) &= (V_{2n+2} - V_2 - V_0 + \sum_{k=0}^{n} V_{2k}) - (-V_0 + \sum_{k=0}^{n} V_{2k}) - (-V_{2n+1} + \sum_{k=0}^{n} V_{2k+1}) \\
&\quad -(-V_{2n} + \sum_{k=0}^{n} V_{2k}) - (-V_{2n+1} - V_{2n-1} + V_{-1} + \sum_{k=0}^{n} V_{2k+1}).
\end{aligned}$$

Similarly, using the recurrence relation

$$V_n = 2V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5}$$

i.e.

$$2V_{n-1} = V_n - V_{n-2} - V_{n-3} - V_{n-4} - V_{n-5}$$

we write the following obvious equations;

$$\begin{aligned}
2V_2 &= V_3 - V_1 - V_0 - V_{-1} - V_{-2} \\
2V_4 &= V_5 - V_3 - V_2 - V_1 - V_0 \\
2V_6 &= V_7 - V_5 - V_4 - V_3 - V_2 \\
&\vdots \\
2V_{2n-2} &= V_{2n-1} - V_{2n-3} - V_{2n-4} - V_{2n-5} - V_{2n-6} \\
2V_{2n} &= V_{2n+1} - V_{2n-1} - V_{2n-2} - V_{2n-3} - V_{2n-4} \\
2V_{2n+2} &= V_{2n+3} - V_{2n+1} - V_{2n} - V_{2n-1} - V_{2n-2} \\
2V_{2n+4} &= V_{2n+5} - V_{2n+3} - V_{2n+2} - V_{2n+1} - V_{2n}
\end{aligned}$$

Now, if we add the above equations by side by, we obtain

$$\begin{aligned}
2(-V_0 + \sum_{k=0}^{n} V_{2k}) &= (-V_1 + \sum_{k=0}^{n} V_{2k+1}) - (-V_{2n+1} + \sum_{k=0}^{n} V_{2k+1}) - (-V_{2n} + \sum_{k=0}^{n} V_{2k}) \\
&\quad -(-V_{2n+1} - V_{2n-1} + V_{-1} + \sum_{k=0}^{n} V_{2k+1}) - (-V_{2n} - V_{2n-2} + V_{-2} + \sum_{k=0}^{n} V_{2k}).
\end{aligned}$$

Then, solving the following system

$$\begin{aligned}
2(-V_1 + \sum_{k=0}^{n} V_{2k+1}) &= (V_{2n+2} - V_2 - V_0 + \sum_{k=0}^{n} V_{2k}) - (-V_0 + \sum_{k=0}^{n} V_{2k}) - (-V_{2n+1} + \sum_{k=0}^{n} V_{2k+1}) \\
&\quad -(-V_{2n} + \sum_{k=0}^{n} V_{2k}) - (-V_{2n+1} - V_{2n-1} + V_{-1} + \sum_{k=0}^{n} V_{2k+1}) \\
2(-V_0 + \sum_{k=0}^{n} V_{2k}) &= (-V_1 + \sum_{k=0}^{n} V_{2k+1}) - (-V_{2n+1} + \sum_{k=0}^{n} V_{2k+1}) - (-V_{2n} + \sum_{k=0}^{n} V_{2k}) \\
&\quad -(-V_{2n+1} - V_{2n-1} + V_{-1} + \sum_{k=0}^{n} V_{2k+1}) - (-V_{2n} - V_{2n-2} + V_{-2} + \sum_{k=0}^{n} V_{2k})
\end{aligned}$$

where

$$\begin{aligned} V_{-1} &= (-V_0 - V_1 - V_2 - 2 \times V_3 + V_4) \\ V_{-2} &= (-V_4 + 3V_3 - V_2) \end{aligned}$$

the required result of (b) and (c) follow.

As special cases of above Theorem, we have the following three Corollaries. First one presents some summing formulas of fifth order Pell numbers.

Corollary 6.2. *For $n \geq 0$ we have the following formulas:*

(a) *(Sum of the fifth order Pell numbers)*

$$\sum_{k=0}^{n} P_k = \frac{1}{5}(P_{n+5} - P_{n+4} - 2P_{n+3} - 3P_{n+2} - 4P_{n+1} - 1).$$

(b) $\sum_{k=0}^{n} P_{2k} = \frac{1}{15}(-P_{2n+2} + 6P_{2n+1} + 7P_{2n} + 3P_{2n-1} + 4P_{2n-2} - 4).$

(c) $\sum_{k=0}^{n} P_{2k+1} = \frac{1}{15}(4P_{2n+2} + 6P_{2n+1} + 2P_{2n} + 3P_{2n-1} - P_{2n-2} + 1).$

Second one presents some summing formulas of fifth order Pell-Lucas numbers.

Corollary 6.3. *For $n \geq 0$ we have the following formulas:*

(a) *(Sum of the fifth order Pell-Lucas numbers)*

$$\sum_{k=0}^{n} Q_k = \frac{1}{5}(Q_{n+5} - Q_{n+4} - 2Q_{n+3} - 3Q_{n+2} - 4Q_{n+1} + 9).$$

(b) $\sum_{k=0}^{n} Q_{2k} = \frac{1}{15}(-Q_{2n+2} + 6Q_{2n+1} + 7Q_{2n} + 3Q_{2n-1} + 4Q_{2n-2} + 41).$

(c) $\sum_{k=0}^{n} Q_{2k+1} = \frac{1}{15}(4Q_{2n+2} + 6Q_{2n+1} + 2Q_{2n} + 3Q_{2n-1} - Q_{2n-2} - 14).$

Third one presents some summing formulas of modified fifth order Pell numbers.

Corollary 6.4. *For $n \geq 0$ we have the following formulas:*

(a) *(Sum of the modified fifth order Pell numbers)*

$$\sum_{k=0}^{n} E_k = \frac{1}{5}(E_{n+5} - E_{n+4} - 2E_{n+3} - 3E_{n+2} - 4E_{n+1}).$$

(b) $\sum_{k=0}^{n} E_{2k} = \frac{1}{15}(-E_{2n+2} + 6E_{2n+1} + 7E_{2n} + 3E_{2n-1} + 4E_{2n-2} - 5).$

(c) $\sum_{k=0}^{n} E_{2k+1} = \frac{1}{15}(4E_{2n+2} + 6E_{2n+1} + 2E_{2n} + 3E_{2n-1} - E_{2n-2} + 5).$

7 Matrices Related with Generalized Fifth-Order Pell numbers

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+4} \\ W_{n+3} \\ W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r_1 & r_2 & r_3 & r_4 & r_5 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_4 \\ W_3 \\ W_2 \\ W_1 \\ W_0 \end{pmatrix} \tag{7.1}$$

For matrix formulation (7.1), see [20]. In fact, Kalman give the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \\ W_{n+3} \\ W_{n+4} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ r_1 & r_2 & r_3 & r_4 & r_5 \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \\ W_4 \end{pmatrix}.$$

We define the square matrix A of order 5 as:

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det M = 1$. From (1.2) we have

$$\begin{pmatrix} V_{n+4} \\ V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \\ V_{n-1} \end{pmatrix}. \tag{7.2}$$

and from (7.1) (or using (7.2) and induction) we have

$$\begin{pmatrix} V_{n+4} \\ V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_4 \\ V_3 \\ V_2 \\ V_1 \\ V_0 \end{pmatrix}.$$

If we take $V = P$ in (7.2) we have

$$\begin{pmatrix} P_{n+4} \\ P_{n+3} \\ P_{n+2} \\ P_{n+1} \\ P_n \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{n+3} \\ P_{n+2} \\ P_{n+1} \\ P_n \\ P_{n-1} \end{pmatrix}. \tag{7.3}$$

We also define

$$B_n = \begin{pmatrix} P_{n+1} & P_n + P_{n-1} + P_{n-2} + P_{n-3} & P_n + P_{n-1} + P_{n-2} & P_n + P_{n-1} & P_n \\ P_n & P_{n-1} + P_{n-2} + P_{n-3} + P_{n-4} & P_{n-1} + P_{n-2} + P_{n-3} & P_{n-1} + P_{n-2} & P_{n-1} \\ P_{n-1} & P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5} & P_{n-2} + P_{n-3} + P_{n-4} & P_{n-2} + P_{n-3} & P_{n-2} \\ P_{n-2} & P_{n-3} + P_{n-4} + P_{n-5} + P_{n-6} & P_{n-3} + P_{n-4} + P_{n-5} & P_{n-3} + P_{n-4} & P_{n-3} \\ P_{n-3} & P_{n-4} + P_{n-5} + P_{n-6} + P_{n-7} & P_{n-4} + P_{n-5} + P_{n-6} & P_{n-4} + P_{n-5} & P_{n-4} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} V_{n+1} & V_n + V_{n-1} + V_{n-2} + V_{n-3} & V_n + V_{n-1} + V_{n-2} & V_n + V_{n-1} & V_n \\ V_n & V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} & V_{n-1} + V_{n-2} + V_{n-3} & V_{n-1} + V_{n-2} & V_{n-1} \\ V_{n-1} & V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5} & V_{n-2} + V_{n-3} + V_{n-4} & V_{n-2} + V_{n-3} & V_{n-2} \\ V_{n-2} & V_{n-3} + V_{n-4} + V_{n-5} + V_{n-6} & V_{n-3} + V_{n-4} + V_{n-5} & V_{n-3} + V_{n-4} & V_{n-3} \\ V_{n-3} & V_{n-4} + V_{n-5} + V_{n-6} + V_{n-7} & V_{n-4} + V_{n-5} + V_{n-6} & V_{n-4} + V_{n-5} & V_{n-4} \end{pmatrix}$$

Theorem 7.1. *For all integer $m, n \geq 0$, we have*

(a) $B_n = A^n$
(b) $C_1 A^n = A^n C_1$
(c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof.

(a) By expanding the vectors on the both sides of (7.3) to 5-colums and multiplying the obtained on the right-hand side by A, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1} B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

(b) Using (a) and definition of C_1, (b) follows.

(c) We have

$$AC_{n-1} = C_n$$

i.e. $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1} C_1$. Now we obtain

$$C_{n+m} = A^{n+m-1} C_1 = A^{n-1} A^m C_1 = A^{n-1} C_1 A^m = C_n B_m$$

and similarly

$$C_{n+m} = B_m C_n.$$

Some properties of A^n matrix can be given as

$$A^n = 2A^{n-1} + A^{n-2} + A^{n-3} + A^{n-4} + A^{n-5}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

for all integer m and n.

Theorem 7.2. *For $m, n \geq 0$ we have*

$$\begin{aligned} V_{n+m} &= V_n P_{m+1} + V_{n-1}(P_m + P_{m-1} + P_{m-2} + P_{m-3}) \\ &\quad + V_{n-2}(P_m + P_{m-1} + P_{m-2}) + V_{n-3}(P_m + P_{m-1}) + V_{n-4} P_m \\ &= V_n P_{m+1} + P_m (V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4}) + P_{m-3} V_{n-1} \\ &\quad + P_{m-1} (V_{n-1} + V_{n-2} + V_{n-3}) + P_{m-2} (V_{n-1} + V_{n-2}) \end{aligned} \tag{7.4}$$

Proof. From the equation $C_{n+m} = C_n B_m = B_m C_n$ we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation we say that an element of C_{n+m} is the product of a row C_n and column B_m.We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and $C_n B_m$. This completes the proof.

Remark 7.1. By induction, it can be proved that for all integers $m, n \leq 0$, (7.4) holds. So for all integers m, n (7.4) is true.

Corollary 7.3. *For all integers m, n, we have*

$$\begin{aligned} P_{n+m} &= P_n P_{m+1} + P_{n-1}(P_m + P_{m-1} + P_{m-2} + P_{m-3}) + P_{n-2}(P_m + P_{m-1} + P_{m-2}) \\ &\quad + P_{n-3}(P_m + P_{m-1}) + P_{n-4} P_m, \\ Q_{n+m} &= Q_n P_{m+1} + Q_{n-1}(P_m + P_{m-1} + P_{m-2} + P_{m-3}) + Q_{n-2}(P_m + P_{m-1} + P_{m-2}) \\ &\quad + Q_{n-3}(P_m + P_{m-1}) + Q_{n-4} P_m, \\ E_{n+m} &= E_n P_{m+1} + E_{n-1}(P_m + P_{m-1} + P_{m-2} + P_{m-3}) + E_{n-2}(P_m + P_{m-1} + P_{m-2}) \\ &\quad + E_{n-3}(P_m + P_{m-1}) + E_{n-4} P_m. \end{aligned}$$

8 Conclusion

In the literature, there have been so many studies of the sequences of numbers and the sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. We introduce the generalized fifth order Pell sequences and we present Binet's formulas, generating functions, Simson formulas, the summation formulas, some identities and matrices for these sequences.

Competing Interests

Author has declared that no competing interests exist.

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