Asian Research Journal of Mathematics

15(3): 1-15, 2019; Article no.ARJOM.52186

ISSN: 2456-477X

Portfolio Selection Strategies with Return Clause in a DC Pension Fund

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Authors' contributions

This work was carried out in collaboration between both authors. Author EEA designed the study, performed the mathematical formulations and wrote the initial draft of the manuscript. Author UOI managed the sensitivity analyses of the study. Both authors managed the writing of the literature, read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2019/v15i330149
Editor(s):
(1) Prof. Wei-Shih Du, Department of Mathematics, National Kaohsiung Normal University, Taiwan.
Reviewers:
(1) Hussin Jose Hejase, Al Maaref University, Lebanon.
(2) Jackson Akpojaro, University of Africa, Toru-Orua, Nigeria.
Complete Peer review History: http://www.sdiarticle4.com/review-history/52186

Original Research Article

Received: 10 August 2019
Accepted: 18 October 2019
Published: 28 October 2019

Abstract

This paper solves the problem faced by a pension fund manager in determining the optimal selection strategies involving four different assets comprising of one risk free asset and three risky assets whose prices are modelled by geometric Brownian motion. Also, a clause mandating the fund managers to return the accumulations with predetermined interest to members who lost their life during the accumulation period is considered. A stochastic optimal control model is formulated comprising of member's monthly contributions, invested funds and the returned contributions. Also, an optimization problem from the extended Hamilton Jacobi Bellman (HJB) equation is established using the game theoretic approach. The explicit solutions of the optimal selection strategies and the efficient frontier are obtained by solving the extended HJB equation using the mean variance utility and separation of variable technique. Furthermore, a sensitivity analysis of the effect of some parameters on the optimal selection strategies is carried out numerically.

Keywords: DC pension fund; extended HJB equation; optimal investment strategies; game theoretic approach; return clause; Geometric Brownian motion.

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2010 Mathematics Subject Classification: 91B16, 90C31, 62P05.

1 Introduction

In our present days, the concept of investment is a very crucial component in managing funds for any financial institution such as banks, pension fund system, insurance company, etc... The study of the optimal selection strategies for pension schemes has drawn attention in the field of optimization theory. In pension, numerous researchers such as Cairns, Blake and Dowd [1], and Giacinto, Gozzi and Federico [2] have tried solving optimization problems for the optimal investment strategies for a pension member with diverse portfolios. There are two major types of pension system; these include the ‘‘Defined Benefit’’ (DB) pension plan and the ‘‘Defined Contribution’’ (DC) pension plan. In the first plan, member’s benefits are fixed in advance by their employers while the contributions at the initial stage are set and afterwards adjusted to stabilize the fund. Different from the DB plan, a DC plan mandate members of the scheme to remit a fix proportion of their income into their ‘‘Retirement Savings Account’’ (RSA). In this plan, members are engaged in investment process and their retirement benefits depend upon the returns of the investments made during the accumulation phase and the expected return can be determined by some factors which include mortality risk, inflation, investment efficiency, government policy etc. Although the DC plan is very transparent, attractive and reliable compared to the DB plan, there is need for members to get acquainted with the financial market on how investments in different assets are carried out. This has led to the study of optimal investment strategies by several authors.

Deelstra, Grasselli and Koehl [3], presented a martingale method to study the related DC plan. Giacinto, Gozzi and Federico [2] and Cairns, Blake and Dowd [1], studied the optimal control strategies during the accumulation phase of the DC plan with diverse utility functions using stochastic optimal control method. Han and Hung [4], investigated the DC pension fund management problem with the inflation risk. Sun, Li and Zeng [5], discussed the optimization problem for the DC plan under a jump-diffusion model. Gao [6], applied the Legendre transform and dual theory to study the DC plan for a pension member’s whole life under the CEV model. Guan and Liang [7], considered the optimal management of the DC plan in a stochastic interest rate and Heston SV model framework. Akpanibah and Samaila [8], studied stochastic strategies of optimal investment for DC pension fund with multiple contributors where they considered the rate of contribution to be stochastic. According to Sheng and Rong [9], the Mean-variance condition was first developed by Markowitz to investigate portfolio selection problem but the optimal investment policies under mean-variance criterion are not time consistent, because the mean-variance condition does not have the iterated expectation property hence the Bellman’s principle of optimality does not hold. It was shown that time consistency of strategies is a major requirement for realistic decision makers. Björk and Murgoci [10], studied the general theory of Markovian time inconsistent stochastic control problems. Bjork, Murgoci and Xunyu [11], studied the portfolio optimization with state-dependent risk aversion in the mean-variance framework.

Recently, He and Liang [12], studied optimal investment strategy for a defined contribution pension scheme with the return of premiums clause under mean-variance utility function; they considered investment in one risk free asset and a risky asset. Li, Rong, Zhao and Yi [13], studied equilibrium investment strategy for DC pension plan with default risk and return of premiums clauses under (CEV) model; they took into consideration investment in one risk free asset ant two risky assets. The optimal time-consistent investment strategy for a DC pension with the return of premiums clauses and annuity contracts was studied by Sheng and Rong [9]. They invested in a risk free asset and a risky asset (stock) and assumed the stock market price to follow Heston volatility model. Osu, Akpanibah and Olunkwa [14], studied optimization problem with return of premium in a DC pension with multiple contributors; they considered a case where the fund manager deals with more than one contributor per time and also investment in two asset where the price of the risky asset followed CEV model. DC pension plan with the return of premium clauses under inflation risk and volatility risk was studied by Wang, Fan and Chang [15]; in their work, they assume investment in a risk free asset, the inflation index bond and the stock whose price is modelled by Heston volatility. Also, Akpanibah, Osu and Ihedioha [16], studied portfolio selection for a DC pension fund with return clauses of premium with predetermined interest rate under mean-variance utility; they considered investment in a risk free and a risky asset and assume the return contributions are predetermined. The optimal asset allocation strategy for a defined contribution pension system with refund clause of premium with predetermined interest under Heston's volatility model was studied by Akpanibah and Osu [17].

Throughout the literatures and to the best of our knowledge no work has been recorded on the optimal selection strategies with return clause that considers investment in four assets comprising of one risk free asset and three risky assets such that the return contributions are with predetermined interest. This forms the basis of this paper.

2 Wealth Formulations

Let us consider a market that is less fractioned, complete and is open continuously for a fixed interval $t \in [0, T]$, where T is defined as the time frame of the accumulation phase. Let $(\Omega, \mathcal{G}, \mathbb{p})$ be a complete probability space where Ω is a real space and $\mathbb{p}$ a probability measure, $\mathcal{G}$ is the filtration and represent the information generated by the Brownian motions $\{\mathcal{B}_1(t), \mathcal{B}_2(t), \mathcal{B}_3(t), \mathcal{B}_4(t), \mathcal{B}_5(t), \mathcal{B}_6(t)\}$ such that the Brownian motions are orthogonal.

Let the following functions $S_0(t)$, $S_1(t)$, $S_2(t)$ and $S_3(t)$ represent the price of the risk-free asset and the three risky assets, and their models are given as follows:

$$\frac{dS_0(t)}{S_0(t)} = \mathcal{r}dt, \tag{2.1}$$

$$\frac{dS_1(t)}{S_1(t)} = (\mathcal{r} + \delta_1)dt + z_1 d\mathcal{B}_1(t). \tag{2.2}$$

$$\frac{dS_2(t)}{S_2(t)} = (\mathcal{r} + \delta_2)dt + m_1 d\mathcal{B}_2(t) + m_2 d\mathcal{B}_3(t) \tag{2.3}$$

$$\frac{dS_3(t)}{S_3(t)} = (\mathcal{r} + \delta_3)dt + \acute{\eta}_1 d\mathcal{B}_4(t) + \acute{\eta}_2 d\mathcal{B}_5(t) + \acute{\eta}_3 d\mathcal{B}_6(t) \tag{2.4}$$

See Muller [18]

$\mathcal{r}, \delta_1, \delta_2, \delta_3, z_1, m_1, m_2, \acute{\eta}_1, \acute{\eta}_2, \acute{\eta}_3$ are constant and $\mathcal{r}$ is the risk-free interest rate, $(\mathcal{r} + \delta_1)$, $(\mathcal{r} + \delta_2)$ and $(\mathcal{r} + \delta_3)$ are the expected instantaneous rate of return of the three risky assets and $z_1, m_1, m_2, \acute{\eta}_1, \acute{\eta}_2, \acute{\eta}_3$ are the instantaneous volatility of the risky assets. let c be the contribution paid to the members' (RSA) at a given time, $\boldsymbol{v}_0$ the initial age of accumulation phase and $\boldsymbol{v}_0 + T$ is the end age, $n\ddot{Y}_{\boldsymbol{v}_0+t}$ is the mortality rate from time t to $t + n$, tc is the premium accumulated at time t, $tcn\ddot{Y}_{\boldsymbol{v}_0+t}$ is the returned premium to the death member's family.

Let $\hbar_1$, $\hbar_2$, and $\hbar_3$ represent the proportion of the members pension wealth invested in the three risky assets such that $\hbar_0 = 1 - \hbar_1 - \hbar_2 - \hbar_3$ is proportion invested in the risk free asset.

Since the surviving members will want to maximize the fund size and at the same time minimize the volatility of the accumulated wealth. There is need for the pension fund managers to formulate an optimal investment problem under the mean-variance criterion as follows:

$$\sup_{\hbar}\{E_{t,w}\mathcal{W}^{\hbar}(T) - Var_{t,w}\mathcal{W}^{\hbar}(T)\} \tag{2.5}$$

Considering the time interval $[t, t + n]$, the differential form associated with the fund size when the remaining wealth is equally shared among the remaining members is given as:

$$\mathcal{W}(t+n)=\begin{pmatrix}\mathcal{W}(t)\left(\hbar_0\frac{S_0(t+n)}{S_0(t)}+\hbar_1\frac{S_1(t+n)}{S_1(t)}+\hbar_2\frac{S_2(t+n)}{S_2(t)}+\hbar_3\frac{S_3(t+n)}{S_3(t)}\right)\\ +cn-tcn\ddot{Y}_{\boldsymbol{v}_0+t}-\hbar_0\mathcal{W}(t)\frac{S_0(t+n)}{S_0(t)}n\ddot{Y}_{\boldsymbol{v}_0+t}\end{pmatrix}\frac{1}{1-n\ddot{Y}_{\boldsymbol{v}_0+t}} \tag{2.6}$$

$$\mathcal{W}(t+n)=\begin{pmatrix}\mathcal{W}(t)\begin{pmatrix}1+(1-\hbar_1-\hbar_2-\hbar_3)\left(\frac{S_0(t+n)-S_0(t)}{S_0(t)}\right)(1-n\ddot{Y}_{\boldsymbol{v}_0+t})\\ +\hbar_1\left(\frac{S_1(t+n)-S_1(t)}{S_1(t)}\right)+\hbar_2\left(\frac{S_2(t+n)-S_2(t)}{S_2(t)}\right)\\ +\hbar_3\left(\frac{S_3(t+n)-S_3(t)}{S_3(t)}\right)\\ -(1-\hbar_1-\hbar_2-\hbar_3)\mathcal{W}(t)n\ddot{Y}_{\boldsymbol{v}_0+t}\end{pmatrix}\\ +cn-tcn\ddot{Y}_{\boldsymbol{v}_0+t}\end{pmatrix}\left(1+\frac{n\ddot{Y}_{\boldsymbol{v}_0+t}}{1-n\ddot{Y}_{\boldsymbol{v}_0+t}}\right) \tag{2.7}$$

The conditional death probability ${}_tq_x=1-{}_tp_x=1-e^{-\int_0^t\pi(\boldsymbol{v}_0+t+s)ds}$, where $\pi(t)$ is the force function of the mortality at time t, and for $n\to 0$,

$$n\ddot{Y}_{\boldsymbol{v}_0+t}=1-\exp\{-\int_0^n\pi(\boldsymbol{v}_0+t+s)ds\}\approx\pi(\boldsymbol{v}_0+t)n+On$$

$$\frac{n\ddot{Y}_{\boldsymbol{v}_0+t}}{1-n\ddot{Y}_{\boldsymbol{v}_0+t}}=\frac{1-\exp\{-\int_0^n\pi(\boldsymbol{v}_0+t+s)ds\}}{\exp\{-\int_0^n\pi(\boldsymbol{v}_0+t+s)ds\}}=\exp\{\int_0^n\pi(\boldsymbol{v}_0+t+s)ds\}-1\approx\pi(\boldsymbol{v}_0+t)n=On$$

$$n\to 0\ ,\ \frac{n\ddot{Y}_{\boldsymbol{v}_0+t}}{1-n\ddot{Y}_{\boldsymbol{v}_0+t}}=\pi(\boldsymbol{v}_0+t)dt\ ,\ n\ddot{Y}_{\boldsymbol{v}_0+t}=\pi(\boldsymbol{v}_0+t)dt\quad cn\to cdt\ ,\ \frac{S_0(t+n)-S_0(t)}{S_0(t)}\to\frac{dS_0(t)}{S_0(t)}\ ,$$
$$\frac{S_1(t+n)-S_1(t)}{S_1(t)}\to\frac{dS_1(t)}{S_1(t)},\frac{S_2(t+n)-S_2(t)}{S_2(t)}\to\frac{dS_2(t)}{S_2(t)},\frac{S_3(t+n)-S_3(t)}{S_3(t)}\to\frac{dS_3(t)}{S_3(t)} \tag{2.8}$$

Substituting (2.8) into (2.7) we have

$$\mathcal{W}(t+n)=\begin{pmatrix}\mathcal{W}(t)\begin{pmatrix}1+(1-\hbar_1-\hbar_2-\hbar_3)(1-\pi(\boldsymbol{v}_0+t)dt)\frac{dS_0(t)}{S_0(t)}\\ +\hbar_1\frac{dS_1(t)}{S_1(t)}+\hbar_2\frac{dS_2(t)}{S_2(t)}+\hbar_3\frac{dS_3(t)}{S_3(t)}\end{pmatrix}\\ +bdt-tb\pi(\boldsymbol{v}_0+t)dt\\ -(1-\hbar_1-\hbar_2-\hbar_3)\mathcal{W}(t)\pi(\boldsymbol{v}_0+t)dt\end{pmatrix}(1+\pi(\boldsymbol{v}_0+t)dt) \tag{2.9}$$

$$d\mathcal{W}(t)=\left\{\begin{matrix}\begin{pmatrix}\mathcal{W}(t)\begin{pmatrix}r+\hbar_1(\delta_1+\frac{1}{\boldsymbol{v}-\boldsymbol{v}_0-t})+\hbar_2(\delta_2+\frac{1}{\boldsymbol{v}-\boldsymbol{v}_0-t})\\ +\hbar_3(\delta_3+\frac{1}{\boldsymbol{v}-\boldsymbol{v}_0-t})\end{pmatrix}\\ +c\left(\frac{\boldsymbol{v}-\boldsymbol{v}_0-2t}{\boldsymbol{v}-\boldsymbol{v}_0-t}\right)\end{pmatrix}dt\\ +\mathcal{W}(t)\begin{pmatrix}\hbar_1z_1d\mathcal{B}_1(t)+\hbar_2\big(m_1d\mathcal{B}_2(t)+m_2d\mathcal{B}_3(t)\big)\\ +\hbar_3\big(\acute{\eta}_1d\mathcal{B}_4(t)+\acute{\eta}_2d\mathcal{B}_5(t)+\acute{\eta}_3d\mathcal{B}_6(t)\big)\end{pmatrix}\end{matrix}\right\}\ \mathcal{W}(0)=w_0 \tag{2.10}$$

Where $\pi(t)$ is the force function and ϑ is the maximal age of the life table and are related as follows

$$\pi(t)=\frac{1}{\boldsymbol{v}+t}\quad 0\le t<\boldsymbol{v} \tag{2.11}$$

3 M-V Utility and Extended HJB equation

In this section, we observed that the mean-variance (MV) control problem in (2.5) is equivalent to the following Markovian time inconsistent stochastic optimal control problem with value function $A(t,w)$. So we employ the game theoretic method cited in Björk and Murgoci [10], to establish the extended Hamilton Jacobi Bellman equation which is a system of non linear PDE.

$$\begin{cases} B(t,w,\hbar) = E_{t,w}[\mathcal{W}^{\hbar}(T)] - \frac{\gamma}{2} Var_{t,w}[\mathcal{W}^{\hbar}(T)] \\ B(t,w,\hbar) = E_{t,w}[\mathcal{W}^{\hbar}(T)] - \frac{\gamma}{2}\left(E_{t,w}[\mathcal{W}^{\hbar}(T)^2] - (E_{t,w}[\mathcal{W}^{\hbar}(T)])^2\right) \\ A(t,w) = \sup_{\hbar} B(t,w,\hbar) \end{cases} \tag{3.1}$$

Following Björk and Murgoci's [10] procedure, the optimal investment strategy $\hbar^*$ satisfies:

$$A(t,w) = \sup_{\hbar} B(t,w,\hbar) \tag{3.2}$$

γ is a constant representing risk aversion coefficient of the members.

Let $u^{\hbar}(t,w) = E_{t,w}[\mathcal{W}^{\hbar}(T)]$, $v^{\hbar}(t,w) = E_{t,w}[\mathcal{W}^{\hbar}(T)^2]$ then

$$A(t,w) = \sup_{\hbar} \ell\left(t,w,u^{\hbar}(t,w),v^{\hbar}(t,w)\right)$$

Where,

$$\ell(t,w,u,v) = u - \frac{\gamma}{2}(v - u^2) \tag{3.3}$$

Theorem 3.1 (verification theorem). If there exist three real functions $\mathcal{L},\mathcal{M},\mathcal{N}$ $[0,T]\times R \to R$ satisfying the following extended Hamilton Jacobi Bellman equation:

$$\begin{cases} \sup_{\hbar} \begin{Bmatrix} \mathcal{L}_t - \ell_t + (\mathcal{L}_w - \ell_w)\left[w\begin{pmatrix} r + \hbar_1(\delta_1 + \frac{1}{v-v_0-t}) \\ +\hbar_2(\delta_2 + \frac{1}{v-v_0-t}) \\ +\hbar_3(\delta_3 + \frac{1}{v-v_0-t}) \end{pmatrix} + c\left(\frac{v-v_0-2t}{v-v_0-t}\right)\right] \\ +\frac{1}{2}(\mathcal{L}_{ww} - U_{ww})(\hbar_1^2 z_1^2 + \hbar_2^2(m_1^2 + m_2^2) + \hbar_3^2(\acute{\eta}_2^2 + \acute{\eta}_2^2 + \acute{\eta}_2^2)) \end{Bmatrix} = 0 \\ \mathcal{L}(T,w) = \ell(t,w,w,w^2) \end{cases} \tag{3.4}$$

Where,

$$U_{ww} = \ell_{ww} + 2\ell_{wu}u_l + 2\ell_{wv}v_l + \ell_{uu}u_w^2 + 2\ell_{uv}u_w v_w + \ell_{vv}v_w^2 = \gamma u_w^2$$

$$\begin{cases} \begin{Bmatrix} \mathcal{M}_t + \mathcal{M}_w\left[w\begin{pmatrix} r + \hbar_1(\delta_1 + \frac{1}{v-v_0-t}) \\ +\hbar_2(\delta_2 + \frac{1}{v-v_0-t}) \\ +\hbar_3(\delta_3 + \frac{1}{v-v_0-t}) \end{pmatrix} + c\left(\frac{v-v_0-2t}{v-v_0-t}\right)\right] \\ +\frac{1}{2}\mathcal{M}_{ww}(\hbar_1^2 z_1^2 + \hbar_2^2(m_1^2 + m_2^2) + \hbar_3^2(\acute{\eta}_2^2 + \acute{\eta}_2^2 + \acute{\eta}_2^2)) \end{Bmatrix} = 0 \\ \mathcal{M}(T,w) = w \end{cases} \tag{3.5}$$

$$\begin{cases} \begin{Bmatrix} \mathcal{N}_t + \mathcal{N}_l\left[w\begin{pmatrix} r + \hbar_1(\delta_1 + \frac{1}{v-v_0-t}) \\ +\hbar_2(\delta_2 + \frac{1}{v-v_0-t}) \\ +\hbar_3(\delta_3 + \frac{1}{v-v_0-t}) \end{pmatrix} + c\left(\frac{v-v_0-2t}{v-v_0-t}\right)\right] \\ +\frac{1}{2}\mathcal{N}_{ll}(\hbar_1^2 z_1^2 + \hbar_2^2(m_1^2 + m_2^2) + \hbar_3^2(\acute{\eta}_2^2 + \acute{\eta}_2^2 + \acute{\eta}_2^2)) \end{Bmatrix} = 0 \\ Y(T,w) = w^2 \end{cases} \tag{3.6}$$

Then $A(t,w) = \mathcal{L}(t,w), u^{\hbar^*} = \mathcal{M}(t,w), v^{\hbar^*} = \mathcal{N}(t,w)$for the optimal selection strategy $\hbar^*$

Proof:

The details of the proof can be found in He and Liang [19], Liang and Huang [20] and Li and Zeng [21].

4. The Optimal Selection Strategies and Efficient Frontier

In this section we used the mean variance utility function (3.3) and separation of variable technique to obtain the optimal selection strategies for the four assets and also the efficient frontier by solving (3.4), (3.5) and (3.6).

Recall that$\ell(t,w,u,v) = u - \frac{\gamma}{2}(v - u^2)$

$$\ell_t = \ell_w = \ell_{ww} = \ell_{wu} = \ell_{wv} = \ell_{uv} = \ell_{vv} = 0,\ \ell_u = 1 + \gamma u,\ \ell_{uu} = \beta, \text{ and } \ell_v = -\frac{\gamma}{2} \tag{4.1}$$

Substituting (4.1) into (3.4) and differentiating (3.4) with respect to $\hbar_1, \hbar_2$ and $\hbar_3$ and solving for $\hbar_1, \hbar_2$ and $\hbar_3$, we have

$$\hbar_1{}^* = -\left[\frac{(\delta_1 + \frac{1}{v - v_0 + t})\mathcal{L}_w}{(\mathcal{L}_{ww} - \gamma\mathcal{M}_w^2) w d_1}\right] \tag{4.2}$$

$$\hbar_2{}^* = -\left[\frac{(\delta_2 + \frac{1}{v - v_0 + t})\mathcal{L}_w}{(\mathcal{L}_{ww} - \gamma\mathcal{M}_w^2) w d_2)}\right] \tag{4.3}$$

$$\hbar_3{}^* = -\left[\frac{(\delta_3 + \frac{1}{v - v_0 + t})\mathcal{L}_w}{(\mathcal{L}_{ww} - \gamma\mathcal{M}_w^2) w d_3}\right] \tag{4.4}$$

Where $d_1 = z_1^2$, $d_2 = m_1^2 + m_2^2$, and $d_3 = ή_2^2 + ή_2^2 + ή_2^2$

Substituting (4.2), (4.3) and (4.4) into (3.4) we have

$$\mathcal{L}_t + \mathcal{L}_w\left[rw + c\left(\frac{v - v_0 - 2t}{v - v_0 - t}\right)\right] - \frac{\mathcal{L}_w^2}{2(\mathcal{L}_{ww} - \gamma\mathcal{M}_w^2)}\left(\frac{(\delta_1 + \frac{1}{v - v_0 + t})^2}{d_1} + \frac{(\delta_2 + \frac{1}{v - v_0 + t})^2}{d_2} + \frac{(\delta_3 + \frac{1}{v - v_0 + t})^2}{d_3}\right) = 0 \tag{4.5}$$

$$\mathcal{M}_t + \mathcal{M}_w\left[rw + c\left(\frac{v - v_0 - 2t}{v - v_0 - t}\right)\right] - \frac{\mathcal{L}_w\mathcal{M}_w}{(\mathcal{L}_{ww} - \gamma\mathcal{M}_w^2)}\left(\frac{(\delta_1 + \frac{1}{v - v_0 + t})^2}{d_1} + \frac{(\delta_2 + \frac{1}{v - v_0 + t})^2}{d_2} + \frac{(\delta_3 + \frac{1}{v - v_0 + t})^2}{d_3}\right) + \frac{\mathcal{M}_{ww}}{2}\left[\frac{\mathcal{L}_w^2}{(\mathcal{L}_{ww} - \gamma\mathcal{M}_w^2)}\left(\frac{(\delta_1 + \frac{1}{v - v_0 + t})^2}{d_1} + \frac{(\delta_2 + \frac{1}{v - v_0 + t})^2}{d_2} + \frac{(\delta_3 + \frac{1}{v - v_0 + t})^2}{d_3}\right)\right] = 0 \tag{4.6}$$

Proposition 4.1

The optimal investment policy for the three assets are given as

$$\hbar_0{}^* = 1 - \hbar_1{}^* - \hbar_2{}^* - \hbar_3{}^* \tag{4.7}$$

$$\hbar_1{}^* = \frac{1}{\gamma w d_1}\left(\delta_1 + \frac{1}{v - v_0 - t}\right)e^{-r(T-t)} \tag{4.8}$$

$$\hbar_2{}^* = \frac{1}{\gamma w d_2}\left(\delta_2 + \frac{1}{v - v_0 - t}\right)e^{-r(T-t)} \tag{4.9}$$

$$\hbar_3{}^{*}=\frac{1}{\gamma w d_3}\left(\delta_3+\frac{\mathbf{1}}{\boldsymbol{v}-\boldsymbol{v}_0-t}\right)e^{-r(T-t)} \tag{4.10}$$

Proof

We assume a solution for $\mathcal{L}(t,w)$ and $\mathcal{M}(t,w)$ as follows:

$$\begin{cases} \mathcal{L}(t,w)=F(t)w+\frac{G(t)}{\gamma} \quad F(T)=1,\ G(T)=0 \\ \mathcal{M}(t,w)=H(t)w+\frac{I(t)}{\gamma} \quad H(T)=1,\ I(T)=0 \\ \mathcal{L}_t=wF_t(t)+\frac{G_t(t)}{\gamma}, \mathcal{L}_w=F(t), \mathcal{L}_{ww}=0,\ \mathcal{M}_t=wH_t(t)+\frac{I_t(t)}{\gamma}, \mathcal{M}_w=H(t), \mathcal{M}_{ww}=0 \end{cases} \tag{4.11}$$

Substituting (4.11) into (4.5), (4.6)

$$\begin{cases} F_t(t)+rF(t)=0 \\ G_t(t)+F(t)c\gamma\left(\frac{\boldsymbol{v}-\boldsymbol{v}_0-2t}{\boldsymbol{v}-\boldsymbol{v}_0-t}\right)+\frac{F^2(t)}{2H^2(t)}\left(\frac{(\delta_1+\frac{1}{\boldsymbol{v}-\boldsymbol{v}_0+t})^2}{d_1}+\frac{(\delta_2+\frac{1}{\boldsymbol{v}-\boldsymbol{v}_0+t})^2}{d_2}+\frac{(\delta_3+\frac{1}{\boldsymbol{v}-\boldsymbol{v}_0+t})^2}{d_3}\right)=0 \end{cases} \tag{4.12}$$

$$\begin{cases} H_t(t)+rH(t)=0 \\ I_t(t)+H(t)c\gamma\left(\frac{\boldsymbol{v}-\boldsymbol{v}_0-2t}{\boldsymbol{v}-\boldsymbol{v}_0-t}\right)+\frac{F(t)}{H(t)}\left(\frac{(\delta_1+\frac{1}{\boldsymbol{v}-\boldsymbol{v}_0+t})^2}{d_1}+\frac{(\delta_2+\frac{1}{\boldsymbol{v}-\boldsymbol{v}_0+t})^2}{d_2}+\frac{(\delta_3+\frac{1}{\boldsymbol{v}-\boldsymbol{v}_0-t})^2}{d_3}\right)=0 \end{cases} \tag{4.13}$$

Solving (4.12) and (4.13), we have

$$F(t)=e^{r(T-t)} \tag{4.14}$$

$$H(t)=e^{r(T-t)} \tag{4.15}$$

$$G(t)=c\gamma\int_t^T\frac{\tau e^{r(T-\tau)}}{\boldsymbol{v}-\boldsymbol{v}_0-\tau}d\tau+\frac{c\gamma}{r}\left(e^{r(T-t)}-1\right)+\frac{1}{2}\begin{bmatrix}\left(\frac{\delta_1{}^2}{d_1}+\frac{\delta_2{}^2}{d_2}+\frac{\delta_3{}^2}{d_3}\right)(T-t)+ \\ \left(\frac{\delta_1}{d_1}+\frac{\delta_2}{d_2}+\frac{\delta_3}{d_3}\right)ln\left(\frac{\boldsymbol{v}-\boldsymbol{v}_0-t}{\boldsymbol{v}-\boldsymbol{v}_0-T}\right)^2+ \\ \left(\frac{1}{d_1}+\frac{1}{d_2}+\frac{1}{d_3}\right)\left(\frac{T-t}{(\boldsymbol{v}-\boldsymbol{v}_0-t)(\boldsymbol{v}-\boldsymbol{v}_0-T)}\right)\end{bmatrix} \tag{4.16}$$

$$I(t)=c\gamma\int_t^T\frac{\tau e^{r(T-\tau)}}{\boldsymbol{v}-\boldsymbol{v}_0-\tau}d\tau+\frac{c\gamma}{r}\left(e^{r(T-t)}-1\right)+\begin{bmatrix}\left(\frac{\delta_1{}^2}{d_1}+\frac{\delta_2{}^2}{d_2}+\frac{\delta_3{}^2}{d_3}\right)(T-t)+ \\ \left(\frac{\delta_1}{d_1}+\frac{\delta_2}{d_2}+\frac{\delta_3}{d_3}\right)ln\left(\frac{\boldsymbol{v}-\boldsymbol{v}_0-t}{\boldsymbol{v}-\boldsymbol{v}_0-T}\right)^2+ \\ \left(\frac{1}{d_1}+\frac{1}{d_2}+\frac{1}{d_3}\right)\left(\frac{T-t}{(\boldsymbol{v}-\boldsymbol{v}_0-t)(\boldsymbol{v}-\boldsymbol{v}_0-T)}\right)\end{bmatrix} \tag{4.17}$$

$$\mathcal{L}(t,w)=\begin{pmatrix} c\int_t^T\frac{\tau e^{r(T-\tau)}}{\boldsymbol{v}-\boldsymbol{v}_0-\tau}d\tau+\frac{c}{r}\left(e^{r(T-t)}-1\right)+we^{r(T-t)} \\ +\frac{1}{2\gamma}\begin{bmatrix}\left(\frac{\delta_1{}^2}{d_1}+\frac{\delta_2{}^2}{d_2}+\frac{\delta_3{}^2}{d_3}\right)(T-t)+ \\ \left(\frac{\delta_1}{d_1}+\frac{\delta_2}{d_2}+\frac{\delta_3}{d_3}\right)ln\left(\frac{\boldsymbol{v}-\boldsymbol{v}_0-t}{\boldsymbol{v}-\boldsymbol{v}_0-T}\right)^2+ \\ \left(\frac{1}{d_1}+\frac{1}{d_2}+\frac{1}{d_3}\right)\left(\frac{T-t}{(\boldsymbol{v}-\boldsymbol{v}_0-t)(\boldsymbol{v}-\boldsymbol{v}_0-T)}\right)\end{bmatrix}\end{pmatrix} \tag{4.18}$$

$$\mathcal{M}(t,w) = \begin{pmatrix} c\int_t^T \frac{\tau e^{r(T-\tau)}}{v-v_0-\tau}d\tau + \frac{c}{r}\left(e^{r(T-t)}-1\right) + we^{r(T-t)} \\ +\frac{1}{\gamma}\begin{bmatrix} \left(\frac{\delta_1{}^2}{d_1}+\frac{\delta_2{}^2}{d_2}+\frac{\delta_3{}^2}{d_3}\right)(T-t) + \\ \left(\frac{\delta_1}{d_1}+\frac{\delta_2}{d_2}+\frac{\delta_3}{d_3}\right)ln\left(\frac{v-v_0-t}{v-v_0-T}\right)^2 + \\ \left(\frac{1}{d_1}+\frac{1}{d_2}+\frac{1}{d_3}\right)\left(\frac{T-t}{(v-v_0-t)(v-v_0-T)}\right) \end{bmatrix} \end{pmatrix} \tag{4.19}$$

From (4.11), we have

$$\mathcal{L}_w = e^{r(T-t)}, \mathcal{L}_{ww} = 0, \mathcal{M}_w = e^{r(T-t)} \tag{4.20}$$

Substituting (4.20) into (4.8), (4.9) and (4.10), we have

$$\hbar_0{}^* = 1 - \hbar_1{}^* - \hbar_2{}^* - \hbar_3{}^*$$

$$\hbar_1{}^* = \frac{1}{\gamma w d_1}\left(\delta_1 + \frac{\mathbf{1}}{v-v_0-t}\right)e^{-r(T-t)}$$

$$\hbar_2{}^* = \frac{1}{\gamma w d_2}\left(\delta_2 + \frac{\mathbf{1}}{v-v_0-t}\right)e^{-r(T-t)}$$

$$\hbar_3{}^* = \frac{1}{\gamma w d_3}\left(\delta_3 + \frac{\mathbf{1}}{v-v_0-t}\right)e^{-r(T-t)}$$

Proposition 4.2

The efficient frontier of the pension fund is given as

$$E_{t,w}[\mathcal{W}^{\hbar^*}(T)] = \begin{pmatrix} c\int_t^T \frac{\tau e^{r(T-\tau)}}{v-v_0-\tau}d\tau + \frac{c}{r}\left(e^{r(T-t)}-1\right) + we^{r(T-t)} \\ + \sqrt{\begin{bmatrix} \left(\frac{\delta_1{}^2}{d_1}+\frac{\delta_2{}^2}{d_2}+\frac{\delta_3{}^2}{d_3}\right)(T-t) + \\ \left(\frac{\delta_1}{d_1}+\frac{\delta_2}{d_2}+\frac{\delta_3}{d_3}\right)ln\left(\frac{v-v_0-t}{v-v_0-T}\right)^2 + \\ \left(\frac{1}{d_1}+\frac{1}{d_2}+\frac{1}{d_3}\right)\left(\frac{T-t}{(v-v_0-t)(v-v_0-T)}\right) \end{bmatrix}\left(Var_{t,w}[\mathcal{W}^{\hbar^*}(T)]\right)} \end{pmatrix} \tag{4.21}$$

Proof

Recall that

$$Var_{t,w}[\mathcal{W}^{\hbar^*}(T)] = E_{t,w}[\mathcal{W}^{\hbar^*}(T)^2] - (E_{t,w}[\mathcal{W}^{\hbar^*}(T)])^2$$

$$Var_{t,w}[L^{\mu^*}(T)] = \frac{2}{\gamma}(\mathcal{M}(t,w) - \mathcal{L}(t,w)) \tag{4.22}$$

Substituting (4.18) and (4.19) into (4.22), we have

$$Var_{t,w}[L^{\mu^*}(T)] = \frac{1}{\gamma^2}\begin{bmatrix} \left(\frac{\delta_1{}^2}{d_1}+\frac{\delta_2{}^2}{d_2}+\frac{\delta_3{}^2}{d_3}\right)(T-t) + \\ \left(\frac{\delta_1}{d_1}+\frac{\delta_2}{d_2}+\frac{\delta_3}{d_3}\right)ln\left(\frac{v-v_0-t}{v-v_0-T}\right)^2 + \\ \left(\frac{1}{d_1}+\frac{1}{d_2}+\frac{1}{d_3}\right)\left(\frac{T-t}{(v-v_0-t)(v-v_0-T)}\right) \end{bmatrix} \tag{4.23}$$

$$\frac{1}{\gamma} = \sqrt{\frac{Var_{t,w}[L^{\mu^*}(T)]]}{\left[\begin{array}{c}\left(\frac{\delta_1{}^2}{d_1}+\frac{\delta_2{}^2}{d_2}+\frac{\delta_3{}^2}{d_3}\right)(T-t)+\\ \left(\frac{\delta_1}{d_1}+\frac{\delta_2}{d_2}+\frac{\delta_3}{d_3}\right)ln\left(\frac{v-v_0-t}{v-v_0-T}\right)^2+\\ \left(\frac{1}{d_1}+\frac{1}{d_2}+\frac{1}{d_3}\right)\left(\frac{T-t}{(v-v_0-t)(v-v_0-T)}\right)\end{array}\right]}} \tag{4.24}$$

$$E_{t,l}[L^{\mu^*}(T)] = \mathcal{M}(t,w) \tag{4.25}$$

Substituting (4.19) into (4.25), we have

$$E_{t,l}[L^{\mu^*}(T) = \left(\begin{array}{c} c\int_t^T \frac{\tau e^{r(T-\tau)}}{v-v_0-\tau}d\tau + \frac{c}{r}\left(e^{r(T-t)}-1\right) + w e^{r(T-t)} \\ +\frac{1}{\gamma}\left[\begin{array}{c}\left(\frac{\delta_1{}^2}{d_1}+\frac{\delta_2{}^2}{d_2}+\frac{\delta_3{}^2}{d_3}\right)(T-t)+\\ \left(\frac{\delta_1}{d_1}+\frac{\delta_2}{d_2}+\frac{\delta_3}{d_3}\right)ln\left(\frac{v-v_0-t}{v-v_0-T}\right)^2+\\ \left(\frac{1}{d_1}+\frac{1}{d_2}+\frac{1}{d_3}\right)\left(\frac{T-t}{(v-v_0-t)(v-v_0-T)}\right)\end{array}\right]\end{array}\right) \tag{4.26}$$

Substitute (4.24) in (4.26), we have:

$$E_{t,w}[\mathcal{W}^{\hbar^*}E_{t,w}[\mathcal{W}^{\hbar^*}(T)] = \left(\begin{array}{c} c\int_t^T \frac{\tau e^{r(T-\tau)}}{v-v_0-\tau}d\tau + \frac{c}{r}\left(e^{r(T-t)}-1\right) + w e^{r(T-t)} \\ \hline + \sqrt{\left[\begin{array}{c}\left(\frac{\delta_1{}^2}{d_1}+\frac{\delta_2{}^2}{d_2}+\frac{\delta_3{}^2}{d_3}\right)(T-t)+\\ \left(\frac{\delta_1}{d_1}+\frac{\delta_2}{d_2}+\frac{\delta_3}{d_3}\right)ln\left(\frac{v-v_0-t}{v-v_0-T}\right)^2+\\ \left(\frac{1}{d_1}+\frac{1}{d_2}+\frac{1}{d_3}\right)\left(\frac{T-t}{(v-v_0-t)(v-v_0-T)}\right)\end{array}\right]}\left(Var_{t,w}[\mathcal{W}^{\hbar^*}(T)]\right)\end{array}\right)$$

5. Sensitivity Analysis

In this section we present numerical simulations of the optimal investment policy with respect to time using the following parameters. $v = 100$; $v_0 = 20$; $\gamma = 0.05$; $r = 0.02$; $c = 1$; $\delta_1 = 0.035$; $\delta_2 = 0.045$; $\delta_3 = 0.055$; $\sigma_1 = 0.85$; $\sigma_2 = 1$; $\sigma_3 = 0.60$; $\sigma_4 = 1.15$, $\sigma_5 = 0.75$; $\sigma_6 = 0.40$; $l = L(t)$; $l_0 = 1$; $T = 40$;; $t = 0:5:20$. Unless otherwise stated

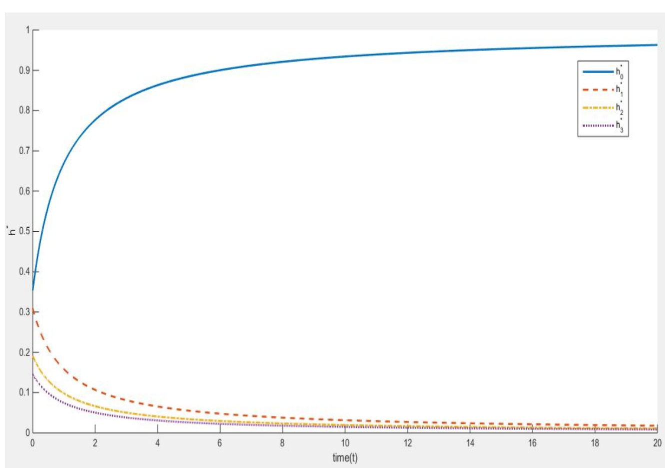

Fig. 1. Time Evolution of $\hbar_0{}^*$, $\hbar_1{}^*$, $\hbar_2{}^*$ & $\hbar_3{}^*$

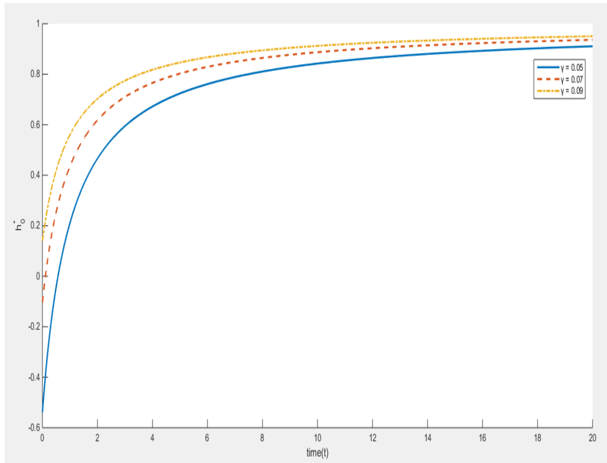

Fig. 2. Evolution of $\hbar_0{}^*$ with different γ

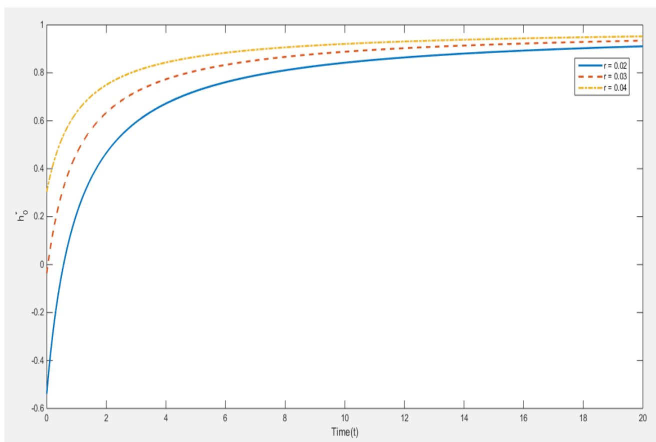

Fig. 3. Evolution of $\hbar_0{}^*$ with different r

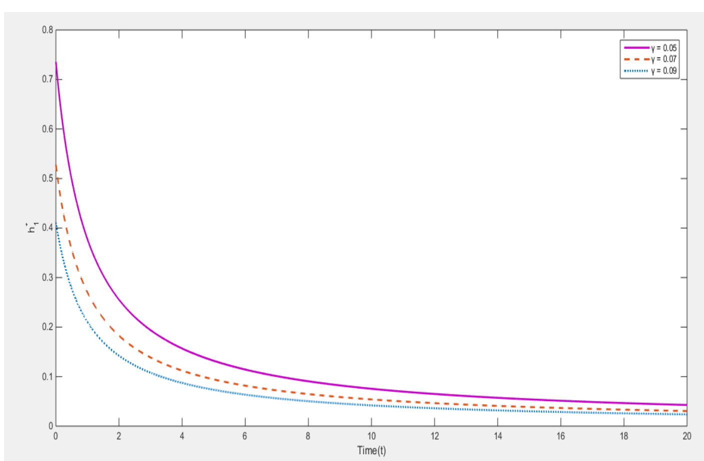

Fig. 4. Evolution of $\hbar_1{}^*$ with different risk averse γ

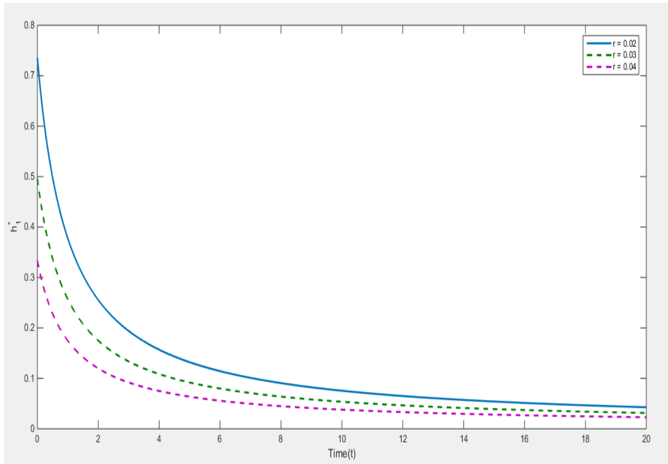

Fig. 5. Evolution of $\hbar_1{}^*$ with different r

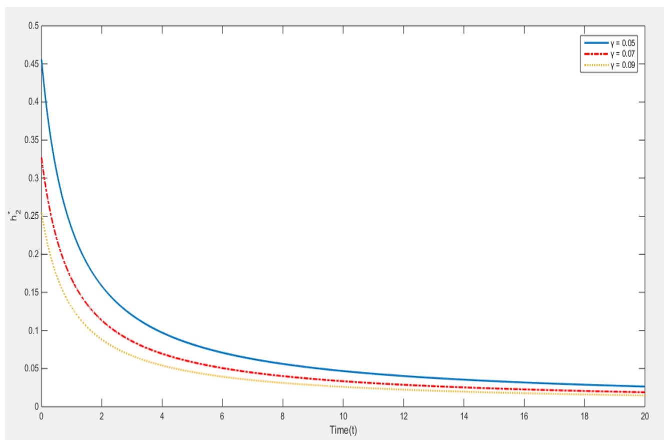

Fig. 6. Evolution of $\hbar_2{}^*$ with different risk averse γ

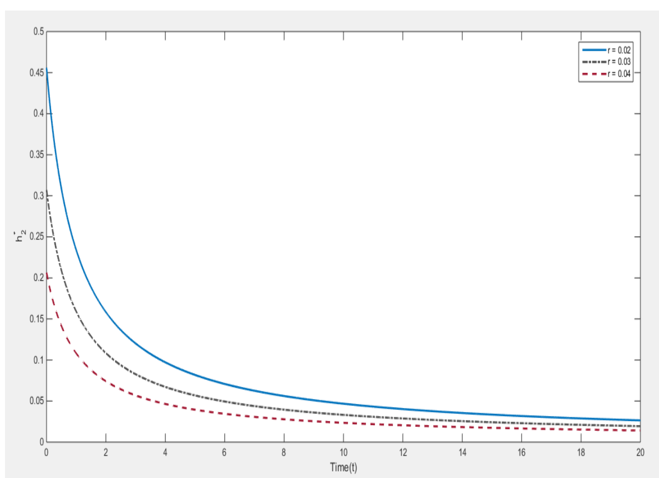

Fig. 7. Evolution of $\hbar_2{}^*$ with different rate r

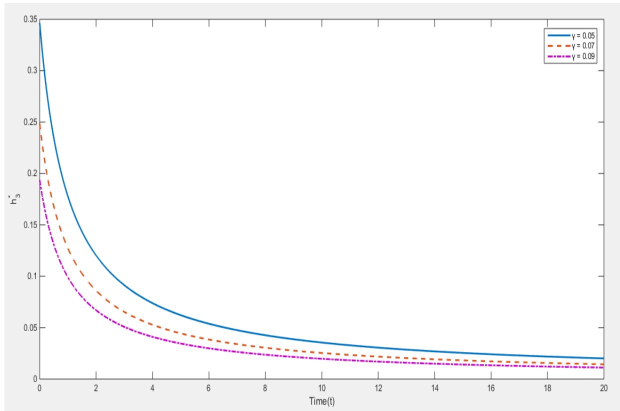

Fig. 8. Evolution of $\hbar_3{}^*$ with different risk averse γ

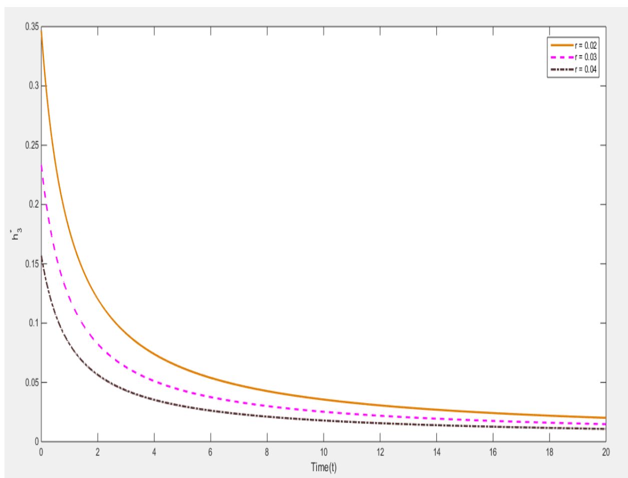

Fig. 9. Evolution of $\hbar_3{}^*$ with different r

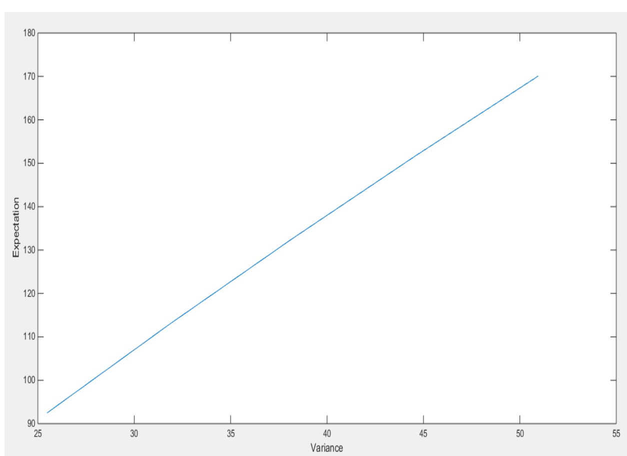

Fig. 10. Time evolution of the expectation with variance

6 Discussion

In Fig. 1, the graph of optimal selection strategies for the four assets with respect to time is presented. It was observed that the optimal selection strategies for the risk free asset increase continuously while that of the risky assets decreases continuously. This is because the optimal fund size was used at the early stage of the investment and as retirement age gets closer, the pension plan manager will want to invest more in risk free asset rather than investing more in the four risky assets. We observed from Figs. 2 and 3, that the optimal selection strategies of the risk free asset increases with risk aversion coefficient of the pension fund member and decreases with the interest rate of the risk free asset. The reason being that members with high risk aversion coefficient are scared of taking risk hence prefer to invest more in risk free assets unlike members with less risk aversion coefficient who prefer taking risk hence a decrease in the fraction invested in risk free asset. Contrarily, when the risk free interest rate is more, members will like to invest where there is more interest and little or no risk hence an increase in investment in risk free asset and a decrease in investment in the risky assets.

From Figs. 4, 6 and 8, we observed that an investor with high risk aversion coefficient will invest less in the three risky assets as retirement age draws closer. In Figs. 5, 7 and 9, we observed that as the predetermined interest rate increases, investment in the three risky assets decreases. Fig. 10, shows that there is a linear relationship between the expectation and the variance; this implies that when more risk is taken by a member, he or she is expecting more returns at the time of retirement and vice versa.

7 Conclusion

In conclusion, the optimal selection strategies for four different assets comprising of one risk free asset and three risky assets whose prices are modelled by geometric Brownian motion was studied when there is a clause mandating the fund managers to return the accumulations with predetermined interest to members who lost their life during the accumulation period. A stochastic optimal control problem was formulated with the help of the actuarial symbol consisting of the invested fund, member's monthly contributions and the returned funds. Next, we established an optimization problem from the extended Hamilton Jacobi Bellman equations and solved the optimization problem for the optimal selection strategies of the four assets and the efficient frontier of the pension members. Furthermore, we present a sensitivity analysis of the optimal selection strategies of the four assets with respect to time.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Cairns AJG, Blake D, Dowd K. Stochastic life styling: Optimal dynamic asset allocation for defined contribution pension plans, Journal of Economic Dynamics & Control. 2006;30(5):843–877.

[2] Giacinto M, Gozzi F, Federico S. Pension funds with a minimum guarantee: A stochastic control approach. Finance and Stochastic. 2011;15:297-342.

[3] Deelstra G, Grasselli M, Koehl PF. Optimal design of the guarantee for defined contribution funds. Journal of Economics Dynamics and Control. 2004;28:2239-2260.

[4] Han N, Hung M. Optimal asset allocation for the DC pension plans under inflation. Insurance: Mathematics and Economics. 2012;51:172-181.

[5] Sun J, Li Z, Zeng Y. Precommitment and equilibrium investment strategies for defined contribution pension plans under a jump-diffusion model. Insurance: Mathematics and Economics. 2016;67:158-172.

[6] Gao J. Optimal portfolios for DC pension plans under a CEV model, Insurance: Mathematics and Economics. 2009;44:479-490.

[7] Guan G, Liang Z. Optimal management of DC pension plan in a stochastic interest rate and stochastic volatility framework. Insurance: Mathematics and Economics. 2014;57:58-66.

[8] Akpanibah EE, Samaila SK**.** Stochastic strategies for optimal investment in a defined contribution (DC) pension fund, International Journal of Applied Science and Mathematical Theory. 2017;3(3):48-55.

[9] Sheng D, Rong X. Optimal time consistent investment strategy for a DC pension with the return of premiums clauses and annuity contracts, Hindawi Publishing Corporation vol; 2014.
Available: http://dx.doi.org/10.1155/2014/862694. 1-13

[10] T. Björk, A. Murgoci. A general theory of Markovian time inconsistent stochastic control problems. Working Paper. Stockholm School of Economics. http://ssrn.com/abstract=1694759. (2010).

[11] Bjork T, Murgoci A, Xunyu Z. "Mean-variance portfolio optimization with state dependent risk aversion," Mathematical Finance. 2012;24(1):1–24.

[12] He L, Liang Z. The optimal investment strategy for the DC plan with the return of premiums clauses in a mean-variance framework, Insurance. 2013;53:643-649.

[13] D Li, X. Rong, H. Zhao, B. Yi. Equilibrium investment strategy for DC pension plan with default risk and return of premiums clauses under CEV model, Insurance. 2017;72:6-20.

[14] Osu BO, Akpanibah EE, Olunkwa O. Mean-Variance Optimization of portfolios with return of premium clauses in a DC pension plan with multiple contributors under constant elasticity of variance model, International journal of mathematical and computational sciences pure and applied researches. 2018;12(5):85-90.

[15] Wang Y, Fan S, Chang H. DC Pension Plan with the Return of Premium Clauses under Inflation Risk and Volatility Risk J. Sys. Sci. & Math. Scis. 2018;38(4):423–437.

[16] Akpanibah EE, Osu BO, Ihedioha SA. On the optimal asset allocation strategy for a defined contribution pension system with refund clause of premium with predetermined interest under Heston's volatility model. J. Nonlinear Sci. Appl. 2020;13(1):53–64.

[17] Akpanibah EE, Osu BO. Optimal Portfolio Selection for a Defined Contribution Pension Fund with Return Clauses of Premium with Predetermined Interest Rate under Mean-variance Utility. Asian Journal of Mathematical Sciences. 2018;2(2):19 –29.

[18] Grant Muller. An optimal investment strategy and multiperiod deposit insurance pricing Model for Commercial Banks. Journal of Applied Mathematics; 2018.
DOI.org/10.1155/2018/8942050

[19] He L, Liang Z. Optimal financing and dividend control of the insurance company with fixed and proportional transaction costs. Insurance: Mathematics & Economics. 2009;44:88–94.

[20] Liang Z, Huang J. Optimal dividend and investing control of an insurance company with higher solvency constraints. Insurance: Mathematics & Economics. 2011;49:501–511.

[21] Li Z, Zeng Y. Optimal time consistent investment and reinsurance policies for mean-variance insurers. Insurance: Mathematics & Economics. 2011;49:145–154.

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