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On Insolvability of the 4-th Hilbert Problem for Hyperbolic Geometries

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Authors' contributions

This work was carried out in collaboration between both authors. Author AS designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author SA managed the analyses of the study and managed the literature searches. Both authors read and approved the final manuscript.

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Abstract

The article proves the insolvability of the 4-th Hilbert Problem for hyperbolic geometries. It has been hypothesized that this fundamental mathematical result (the insolvability of the 4-th Hilbert Problem) holds for other types of non-Euclidean geometry (*geometry* of *Riemann* (elliptic geometry), *non-Archimedean geometry*, and *Minkowski geometry*). The ancient Golden Section, described in Euclid's *Elements* (Proposition II.11) and the following from it Mathematics of Harmony, as a new direction in geometry, are the main mathematical apparatus for this fundamental result. By the way, this solution is reminiscent of the insolvability of the 10-th Hilbert Problem for Diophantine equations in integers. This outstanding mathematical result was obtained by the talented Russian mathematician Yuri Matiyasevich in 1970, by using Fibonacci numbers, introduced in 1202 by the famous Italian mathematician Leonardo from Pisa (by the nickname Fibonacci), and the new theorems in Fibonacci numbers theory, proved by the outstanding Russian mathematician Nikolay Vorobyev and described by him in the third edition of his book "Fibonacci numbers".

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1 Introduction: Hilbert's Problems

David Hilbert is a German mathematician, who made a significant contribution to the development of many areas of mathematics. In 1900, from 6 to 12 August 1900, the II International Congress of Mathematicians was held in Paris. At this Congress, Hilbert presented his report "Mathematical Problems", in which he proposed his famous twenty-three problems of mathematics.

Currently, the 11 problems among the 23 problems have been solved. The 6 problems have been partially solved. For the two problems, mathematicians have no consensus, the 4-th and 23 problems are formulated too vaguely to judge whether they are solved or not (for more details see [1-4]).

2 The 4-th Hilbert Problem

In the list of the 23 Hilbert Problems, the 4-th Problem is formulated as follows: "*Enumerate the metrics, in which the lines are geodesic*."

The 4-th problem consists in studying geometries, "near" in a certain sense to Euclidean geometry. Hilbert explains the meaning of the 4-th Problem as follows:

"A more general question, arising in this case, is the following: is it possible from other fruitful points of view to construct geometries that with the same right could be considered closest to ordinary Euclidean geometry ..."

Under those nearest to Euclidean geometry, Hilbert indicated *geometry of Lobachevsky* (hyperbolic geometry), *geometry of Riemann* (elliptic geometry), *non-Archimedean geometry*, and *geometry* of *Minkowski.*

3 The Fifth Postulate of Lobachevsky

This postulate sounds as follows:

> *"If a straight line and a point lie on a plane, then at least two straight lines can be drawn through this point; they do not intersect each to other"*

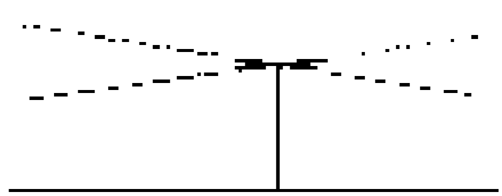

Fig. 1. Illustration of the fifth postulate of Lobachevsky

Thus, the Lobachevsky hyperbolic geometry admits that on the same plane there can be several straight lines at once that do not intersect each other. But in the Euclidean geometry, through a point that does not belong to this straight line, we can draw one and only one straight line; these stright lines do not intersect .

On February 11, 1826 at the Kazan University the session of the Physics and Mathematics Section was hold. On this session Lobachevsky made a speech on the discovery of new geometry. During 1829-30 he published the five articles with the title "On the Principles of Geometry" in the journal "Kazan Bulletin" (the Imperial Kazan University) (see http://www.raruss.ru/russian-thought/597-lobachevsky.html).

The work "On the Principles of Geometry" was, at Lobachevsky's request, presented in 1832 by the Council of Kazan University to the Academy of Sciences. The Academy's conference had decided to give Lobachevsky's work to academician M.V. Ostrogradsky, the acknowledged leader of the Russian Empire mathematicians. In his review M.V. Ostrogradsky wrote the following:

> *"The author apparently set himself the goal of writing in such a way that he could not be understood. He achieved this goal. Everything that I understood in Lobachevsky's geometry is lower than mediocre. Lobachevsky's work does not deserve the attention of the Academy.*
> (See: http://dfgm.math.msu.su/files/encyclopedia/Lobachevski220.pdf)

Among other colleagues, almost no one to support Lobachevsky's geometry; moreover, misunderstanding and ignorant ridicule grew. Trying to find understanding abroad, in 1837 Lobachevsky published his article "Imaginary Geometry" in the German journal "Krell".

Lobachevsky's geometry was widely recognized and widely adopted only 12 years after his death, when it became clear that a scientific theory, built on the basis of a certain axiom system, is considered only then full completed, when its system of axioms satisfies to three conditions: *independence*, *consistency* and *completeness.* Lobachevsky's geometry did satisfy to these three properties.

It is important to note that the Hungarian mathematician Janos Bolyai also came to similar conclusions about hyperbolic geometry, and the famous German mathematician Karl Friedrich Gauss (1777–1855) came to such conclusions even earlier. Gauss generally refrained from publishing on this topic, and Bolyai's works didn't attract attention, and he soon abandoned this topic.

As a result, Nikolay Lobachevsky remained as the unique most consistent propagandist of new geometry.

4 Particular Solutions to the 4-th Hilbert Problem

The dissertation of German mathematician Georg Hamel [5], defended in 1901 under Hilbert's supervision, was the first contribution to the solution of the 4-th Hilbert Problem.

As the American geometer Herbert Busemann (1905 –1994) indicated in the article [6], *"Hamel's work, of course, did not exhaust everything that can be said about Fourth Hilbert Problem, other approaches to which were repeatedly proposed later".*

Let's dwell in more detail on the important contribution to the solution of this problem, made by the outstanding Soviet mathematician A.V. Pogorelov [7]. The summary to Pogorelov's book [7] states the following:

> *"The book contains a solution to the well-known Hilbert's problem on the definition of all, up to isomorphism, realizations of the systems of axioms of classical geometries (Euclidean, Lobachevsky, elliptic), if we omit the congruence axioms, containing the concept of angle, and we supplement of these systems with the axiom of "triangle inequality": the length of any side of the triangle always does not exceed the sum of the lengths of its two other sides".*

A detailed analysis of all modern works, devoted to the 4-th Hilbert Problem, is given in Aranson's article **"Again on the 4th Hilbert Problem**" [8]. In this article, Aranson gives the detailed analysis of the solution of the 4-th Hilbert Problem, made by Alexey Pogorelov.

In Aranson's opinion, if Pogorelov replaces the axioms of congruence of angles by the axiom of "triangle inequality", then for every of the next geometries: *Euclidean geometry* (Euclid), *hyperbolic geometry* (Lobachevsky), *elliptic geometry* (Riemann), when we realize these geometries, the axiom of the congruence of angles becomes the theorem of the congruence of angles. Otherwise, Pogorelov's system of axioms cannot satisfy to three conditions: *independence, consistency* and *completeness*. Therefore, after the actual proof of this newly emerged theorem on the congruence of angles, when we realize Pogorelov's axioms, all previous systems of axioms for *Euclidean*, *Lobachevsky* and *Riemann* geometries are automatically restored.

In Aranson's opinion, this is Pogorelov's contribution to the 4-th Hilbert Problem, and, therefore, what all he did, is the *particular*, but *not the complete solution* to the 4-th Hilbert Problem.

5. Authors' Particular Solution to the 4-th Hilbert Problem, Based on the Hyperbolic Fibonacci λ-Functions

Definitions and sourses. A new stage in the solution of the 4-th Hilbert Problem begins with Alexey Stakhov's book **"The Mathematics of Harmony. From Euclid to Contemporary Mathematics and Computer Science" (World Scientific, 2009)** [9] and with the original books of the Argentinean mathematician Vera de Spinadel [10], of the French engineer and inventor Midhat Gazale [11], of the American mathematician Jay Kappraff [12], of the American philosopher Scott Olsen [13], of the Armenian philosopher Hrant Arakelian [14] and finally with the book of The Prince of Wales with Tony Juniper and Ian Skelly **"Harmony. A New Way of Looking at Our World"** [15].

The purpose of this section is the particular solution to the 4-th Hilbert Problem, which consists in constructing an infinite set of new geometries, "near" to Lobachevsky's geometry, but with other metric properties. The mathematical basis for such solution is the creation by the authors of the general algorithm: the authors used for this purpose Stakhov's book "The Mathematics of Harmony. From Euclid to Contemporary Mathematics and Computer Science" [9]. The Mathematics of Harmony and the 4-th Hilbert Problem is the way to the Harmonic Hyperbolic and Spherical Words of Nature [16,17]. The «Golden» Non-Euclidean Geometry [16,17], the so-called "metallic" proportions by Vera Spinadel [10], Stakhov and Rozin's symmetrical hyperbolic functions [19] and Stakhov's hyperbolic Fibonacci λ-functions [19] were used by the authors in the study of the 4-th Hilbert Problem.

The "metallic" proportions [10], indicated by the symbol Φ_λ, are given by the Spinadel's formula $\Phi_\lambda = \frac{\lambda + \sqrt{4+\lambda^2}}{2}$ [10]**.** For all values of $\lambda \in (-\infty,+\infty)$, the function Φ_λ>0. For $\lambda \to -\infty$ $\Phi_\lambda \to 0$, for $\lambda = 0$ Φ_λ=1, for $\lambda \to +\infty$ $\Phi_\lambda \to +\infty$. For $\lambda = 1$, the formula $\Phi_\lambda = \frac{\lambda + \sqrt{4+\lambda^2}}{2}$ is reduced to the classical golden proportion $\Phi = \frac{1+\sqrt{5}}{2}$, that is, the Spinadel's metallic proportion Φ_λ is a generalization of the formula for the golden proportion $\Phi = \frac{1+\sqrt{5}}{2}$.

The hyperbolic Fibonacci λ-sine and λ-cosine [19] have the following forms, respectively:

$$sF_\lambda(x) = \frac{\Phi_\lambda^x - \Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}} = \frac{2}{\sqrt{4+\lambda^2}} \operatorname{sh}\,[x\ln(\Phi_\lambda)], \qquad cF_\lambda(x) = \frac{\Phi_\lambda^x + \Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}} = \frac{2}{\sqrt{4+\lambda^2}} \operatorname{ch}[x\ln(\Phi_\lambda)].$$

For the case λ =1, the hyperbolic Fibonacci λ -sine and λ -cosine are reduced to the symmetrical hyperbolic Fibonacci sine $sF(x)$ and cosine $cF(x)$ [18], respectively.

Lobachevsky metric and Lobachevsky classical metric. Denote by

Π^+: $\{0 < u < +\infty, -\infty < v < +\infty\}$ the half-plane on the plane Π**:** $\{-\infty < u < +\infty, -\infty < v < +\infty\}$.

We equip the half-plane Π^+ with the metric, which, by following to the terminology [19], is called the *Lobachevsky metric*. This metric has the form $(ds)^2 = R^2\left[(du)^2 + \mathrm{sh}^2(u)(dv)^2\right]$, where ds is the length element. The coefficient $R > 0$ is called the *radius of curvature* of this metric, and the *Gaussian curvature* of this metric is $K = -\dfrac{1}{R^2} < 0$.

The concepts of Gaussian curvature and radius of curvature of a metric [20]. Classical Lobachevsky's metric is given on all the plane

Π'**:** $\{-\infty < u' < +\infty, -\infty < v' < +\infty\}$ and has the form $(ds')^2 = (du')^2 + \mathrm{ch}^2\left(\dfrac{1}{R}u'\right)(dv')^2$, where R >0 is the radius of curvature of the classical Lobachevsky metric [20], [21] . There is shown in [21], that for the given radius R=1 the *classical Lobachevsky metric* is *isometric* to the *Lobachevsky metric* (the concept of *isometry* will be given below). In addition, according to the formulas, indicated below, it is easy to show that *Gaussian curvature* for the classical Lobachevsky metric with $R' = R$ is also equal to $K = -\dfrac{1}{R^2} < 0$.

Isometric displaying and isometry [22]. Let f be a *displaying* from the metric space A to the metric space A', that is, $f(A) \in A'$. If the *displaying* of f preserves the distance between the points, that is, from the conditions $\{x,y\} \in A$ and $\{x'=f(x), y'=f(y)\} \in A'$ it follows $\rho_\lambda(x,y) = \rho_{\lambda'}(x'=f(x), y'=f(y))$, then the *displaying* $f: A \to A'$ is called *isometric.*

The *isometric displaying* $f: A \to A'$ is called *isometry* of the metric space A to the metric space A', and the spaces A and A' are *isometric*. The isometric spaces A and A' are called *homeomorphic*, if the *displaying* $f: A \to A'$ is a *one-to-one and mutually continuous displaying*.

Isometric surfaces [23]. Isometric surfaces in *Euclidean* or *Riemannian* spaces are such surfaces, between which there is the *isometry* with respect to internal metrics, induced on them by the metric of the ambient space.

When we compare on the *isometry* (preservation of lengths) of two internal metrics on surfaces, the following property is important (*Gaussian theorem*) [24]:

"*For the displayings that preserve length (isometry), the Gaussian curvature at the corresponding points is the same, that is*, $K = K'$

There is explained in [24] that if the *displaying* is *isometric* (preserves the lengths of the curves), then it is also *conformal* (preserves angles) and *equiareal* (preserves areas). Conversely: if the *displaying* is *conformal* and *equiareal*, then it is *isometric*.

But then it follows from the *Gauss theorem* that the *displaying* at the corresponding points their Gaussian curvatures K и K' are *inconsistent* ($K \neq K'$), then this *displaying* are *nonisometric* (don't preserves the lengths). Therefore, when $K \neq K'$, by virtue of the Gauss theorem and the above remark about *isometry* [23], we get that if the *displaying* is *nonisometric* (does not save length), then, a priori, only the following situations are possible:

1) either the *displaying* is *nonconformal* (does not preserve angles) and *nonequiareal* (does not preserve areas);
2) either the *displaying* is *nonconformal* (does not preserve angles), but is *equiareal* (save areas);
3) either the *displaying* is *conformal* (preserve angles), is *nonequiareal* (does not preserve areas).

The first quadratic form. Let us give the necessary known facts of differential geometry of surfaces. Let the surface M^2 be given in parametric form:

M^2: $x = x(u, v), y = y(u, v), z = z(u, v)$,

where (u, v) belong to any area D of surface parameters.

The first quadratic form (that is, the differential of arc length) in this case looks as follows:

$$(ds)^2 = E(du)^2 + 2Fdudv + G(dv)^2 \quad ,$$

where

$E = E(u, v) > 0$, $F = F(u, v)$, $G = G(u, v) > 0$, $EG - F^2 > 0$.

Let the surfaces of M^2 : $\{x=x(u,v), y=y(u,v), z=z(u,v)\}$ and

M'^2: $\{x'(u,v), y'=y'(u,v), z'=z'(u,v)\}$ are given in one and the same area

Π: $\{ 0 < u < +\infty, -\infty < v < +\infty \}$ for the parameters u, v (possibly after changing the parameters).

The table below presents the necessary and sufficient condition on the metric of a general form, induced from space, when the indicated metric properties under a one-to-one *displaying* $f : M^2 \Rightarrow M'^2$ of the surface M^2 on the surface M'^2 remain unchanged.

Table of the comparison of metric properties [24]

Displayings	Necessary and sufficient conditions imposed on the metric form
Preserving lengths (isometric)	$E = E'$, $F = F'$, $G = G'$
Preserving angles (conformal)	$E = \lambda_0 E'$, $F = \lambda_0 F'$, $G = \lambda_0 G'$, $\lambda_0 > 0$
Preserving areas (equiarial)	$EG - (F)^2 = E'G' - (F')^2$

In the given Table E, F, G and E', F', G' are coefficients of the metric forms,

corresponding to the points $M(x, y, z) \in M^2$ and $M'(x', y', z') \in M'^2$. These metric forms have the following forms:

$(ds)^2 = E(du)^2 + 2Fdudv + G(dv)^2$, where $E = E(u, v) > 0$, $F = F(u, v)$, $G = G(u, v) > 0$,

$EG - F^2 > 0$,

$(ds')^2 = E'(du)^2 + 2F'dudv + G'(dv)^2$, where $E' = E'(u, v) > 0, F' = F'(u, v)$,

$G' = G'(u, v) > 0$,

$E'G' - F'^2 > 0$.

The construction of new metrics, "near" to the Lobachevsky metric, having other metric properties. As the basic metric, we will consider the *Lobachevsky metric*:

$(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2$,

given in the half-plane $\Pi^+ : \{0 < u < +\infty, -\infty < v < +\infty\}$. The coefficients of the *basic Lobachevsky metric* have the following form: E=1, F=0, $G = \mathrm{sh}^2(u) > 0$. This metric has the radius of curvature R=1 and, therefore, the Gaussian curvature $K = -\frac{1}{R^2} = -1$.

In this situation, the *Lobachevsky basic metric* is realised on the pseudo sphere

M^2: $Z^2 - X^2 - Y^2 = 1$ in the three-dimensional Minkowski space (X, Y, Z), endowed with *Minkowski metric* $(dl)^2 = (dZ)^2 - (dX)^2 - (dY)^2$ for parameterization

$X = \mathrm{sh}(u)\cos(v)$, $Y = \mathrm{sh}(u)\sin(v)$, $Z = \mathrm{ch}(u)$.

As metrics, which will be compared with the *basic Lobachevsky metric*, in order to study the discrepancy of metric properties with the *basic Lobachevsky metric*, we will consider the types of metrics, set for any values of the coefficients $\{\alpha \neq 0, \alpha \neq \pm 1\}$ and at the values of the parameters (u,v), on the half-plane Π:$\{0 < u < +\infty, -\infty < v < +\infty\}$, as the *basic Lobachevsky metric.* We will name them as *comparative metrics*. A view of these *comparative metrics* will be indicated below. More complex types of *comparative metrics* are given in authors' monograph [14].

Further, besides the first approach of comparing the main Lobachevsky metric $(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2$ with comparative metrics in terms of ordinary hyperbolic functions - hyperbolic sine sh or hyperbolic cosine ch, depending on the parameter u of the coefficient α, the second approach is also used.

This second approach is based on the use of hyperbolic Fibonacci $\lambda-$ functions, namely the hyperbolic Fibonacci $\lambda-$sine sF_λ or hyperbolic Fibonacci $\lambda-$cosine cF_λ, dependent on the parameter u and the coefficient λ (see [21]).

The connection between the hyperbolic functions in the first approach and the second approach is carried out by replacing the coefficient α by the coefficient λ according to the formula $\lambda = 2\,\mathrm{sh}(\alpha) \Leftrightarrow \alpha = \mathrm{arsh}(\frac{\lambda}{2}) \equiv \ln(\Phi_\lambda)$, where the function $\Phi_\lambda = \frac{\lambda + \sqrt{4+\lambda^2}}{2}$ in [10] is named *"metallic proportion"*.

For the cases $\lambda = 1,2,3,4$, the following special terms to the values of the function Φ_λ: *golden, silver, bronze,* and *copper proportions* are assigned in [10].

In order to be able to talk about the "proximity" of these comparative metrics to the main metric of Lobachevsky $(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2$, pre-entered, two functions $\rho = \rho(\alpha)$, $\overline{\rho} = \overline{\rho}(\lambda)$ from the above factors $\alpha, \lambda \in \{-\infty, +\infty\}$ of the kind $\rho = |\alpha^2 - 1| \geq 0$, $\overline{\rho} = |[\ln^2(\Phi_\lambda)] - 1| \geq 0$. When replacing α =

$\ln(\Phi_\lambda)$, we get $\overline{\rho}(\lambda) = \rho(\alpha = \ln(\Phi_\lambda)) = \left|[\ln^2(\Phi_\lambda)] - 1\right|$. The above functions ρ and $\overline{\rho}$ are named the "distance" of the *comparative metrics* from the *main Lobachevsky metric*, respectively, in the first and second approaches.

When $\alpha = \pm 1$ (the first approach) and, accordingly, when $\lambda = 2\,\mathrm{sh}((\alpha = \pm 1) = \pm\, 2.3504$ (the second approach) we get $\rho = 0$, $\overline{\rho} = 0$ (coincidence of the *comparative metrics* with the *Lobachevsky metric*). When $\alpha \neq \pm 1$, $\lambda \neq \pm 2.3504$ we get $\rho > 0$, $\overline{\rho} > 0$ (*non-coincidence* of the *comparative metrics* with *Lobachevsky metric*).

6 Particular Solution of the 4-th Hilbert Problem for Hyperbolic Geometries

Theorem 1. Among the infinite set Ω of the hyperbolic metrics of negative Gaussian curvatures there is an infinite subset Ω_0 of the comparative metrics with the same negative Gaussian curvature $K = -1$, and the different negative Gaussian curvatures $K \neq -1$, which are arbitrarily near to the Lobachevsky Gaussian curvature metric $K = -1$. There is a single general algorithm of the detection of the comparative metric properties, based on the Taylor power series decomposition; this algorithm are satisfying to the following conditions:

1) when comparing any comparative metric with the Lobachevsky metric, *nonisometry*, *non-conformity* and *nonequivariality* take place;
2) when comparing two pairs of comparative metrics with each other, *nonisometry*, *nonconformity* and *nonequivariality* take place.

The first type of comparison of metrics with $\{u > 0,\ v \in (-\infty, +\infty)\}$.

The *basic Lobachevsky metric* has the form: $(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2$ and its Gaussian curvature is equal $K = -1$). The *comparative metric* of the first type has the form: $(ds')^2 = \alpha^2 (du)^2 + \mathrm{sh}^2(\alpha u)(dv)^2$, $\{\alpha \neq 0, \alpha \neq \pm 1\}$ and its Gaussian curvature is equal $K' = -1$; in this case: $K' = K = -1$.

Representation of comparisons of the metrics of the first type in terms of hyperbolic Fibonacci λ-functions. Let's assume that we have:

$\alpha = \ln(\Phi_\lambda) \Leftrightarrow \lambda = 2\,\mathrm{sh}(\alpha), \{\alpha \neq 0, \alpha \neq \pm 1\} \Leftrightarrow \{\lambda \neq 0, \lambda \neq \pm 2.3504\}$. Then, we get the metric $(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2$ with the Gaussian curvature $K = -1$; and let's consider the next example of the metric

$$(ds')^2 = \ln^2(\Phi_\lambda)(du)^2 + \mathrm{sh}^2[u \bullet \ln(\Phi_\lambda)]\,(dv)^2 = \ln^2(\Phi_\lambda)(du)^2 + \frac{4+\lambda^2}{4}\, sF_\lambda^2(u)\ (dv)^2$$

with the Gaussian curvature $K' = -1$, that is, $K' = K = -1$.

The second type of comparison of the metrics with $\{u > 0,\ v \in (-\infty, +\infty)\}$.

The *basic Lobachevsky metric* has the form: $(ds)^2 = (du)^2 + \text{sh}^2(u)(dv)^2$ and its the Gaussian creature is equal: $K = -1$. The *comparative metric* of the second type has the following form: $(ds')^2 = (du)^2 + \frac{1}{\alpha^2}$ $\text{sh}^2(\alpha u)(dv)^2$, $\{\alpha \neq 0, \alpha \neq \pm 1\}$ and its Gaussian curvature is equal: $K' = -\alpha^2 < 0$); here we have: $K' \neq K$.

Representation of the second type of comparison of metrics in terms of hyperbolic Fibonacci λ-functions. Let's assume the following:

$\alpha = \ln(\Phi_\lambda) \Leftrightarrow \lambda = 2\,\text{sh}(\alpha), \{\alpha \neq 0, \alpha \neq \pm 1\} \Leftrightarrow \{\lambda \neq 0, \lambda \neq \pm 2.3504\}$. Then, for this case we get the metric: $(ds)^2 = (du)^2 + \text{sh}^2(u)(dv)^2$ with the Gaussian curvature $K = -1$. By using the hyperbolic Fibonacci λ-functions, we can represent the example of the second type of the *comparative metric* as follows:

$$(ds')^2 = (du)^2 + \frac{1}{\ln^2(\Phi_\lambda)}\text{sh}^2[u \bullet \ln(\Phi_\lambda)](dv)^2 = (du)^2 + \frac{4+\lambda^2}{\ln^2(\Phi_\lambda^2)}\, sF_\lambda^2(u)\ (dv)^2 .$$

The Gaussian curvature has the form: $K' = -\ln^2(\Phi_\lambda) < 0$ and consequently $K' \neq K$.

The *basic Lobachevsly metric* has the form: $(ds)^2 = (du)^2 + \text{sh}^2(u)(dv)$ and the geodesic curvature $K = -1$. Then, for the condition $\Phi_{\lambda_0} = e$, were $e \approx 2.71828$, we get: $K'(\lambda_0) = K = -1$. For this case we have: $\lambda_0 = \pm(e - \frac{1}{e}) = \pm\, 2.3504$.

The question of constructing other geometries with negative Gaussian curvatures, nearest to the Lobachevsky geometry, but having different metric properties in comparison with it (*nonisometric, nonconformal, noninequal*), is fundamental. Such geometries, nearest to *Lobachevsky's geometry*, are also the nearest geometries (in Hilbert sense) and to *Euclidean geometry*.

The Gaussian curvature of comparative metrics of the first kind. Let the *comparative metric* of the first kind be given:

$$(ds')^2 = \alpha^2(du)^2 + \text{sh}^2(\alpha u)(dv)^2, \text{ where } \{\alpha \neq 0, \alpha \neq \pm 1, u>0, -\infty < v < +\infty\}.$$

The coefficients of this metric are as follows:

$E' = \alpha^2 > 0$, $F' = 0$, $G' = \text{sh}^2(\alpha u) > 0$.

The Gaussian curvature for this case is equal: $K' = -1 < 0$. But then the radius of curvature R' of the first *comparative metric* $(ds')^2 = \alpha^2(du)^2 + \text{sh}^2(\alpha u)(dv)^2$ is equal $R' = \frac{1}{\sqrt{-K'}} = 1$.

The first *comparative metric* is realized for parameterization:

$X' = \text{sh}(\alpha u)\cos(v)$, $Y' = \text{sh}(\alpha u)\sin(v')$, $Z' = \text{ch}(\alpha u)$

on the pseudo-sphere $M'^2 : Z'^2 - X'^2 - Y'^2 = 1$, $Z' \geq 1$ in three-dimensional Minkowski space (X, Y, Z), endowed with the Minkowski metric $(dl)^2 = (dZ)^2 - (dX)^2 - (dY)^2$.

On the pseudo-sphere $M'^{2}: Z'^2 - X'^2 - Y'^2 = 1$, $Z' \geq 1$ with the above parameterization of the *comparative metric* we get the relation:

$$-[(dZ')^2 - (dX')^2 - (dY')^2] = (ds')^2 = \alpha^2 (du)^2 + \operatorname{sh}^2(\alpha u)(dv)^2 .$$

Thus, with $\rho = |\alpha^2 - 1| > 0$, the *comparative metric* of the first type

$$(ds')^2 = \alpha^2 (du)^2 + \operatorname{sh}^2(\alpha u)(dv)^2, \{\alpha \neq 0, \alpha \neq \pm 1, u>0, -\infty < v < +\infty\}$$

has the same geodesic curvature $K' = -1$, as the geodesic curvature $K = -1$ of the *basic Lobachevsky metric* $(ds)^2 = (du)^2 + \operatorname{sh}^2(u)(dv)^2$. The carrier of these two metrics turned out to be the same pseudo-sphere: $Z^2 - X^2 - Y^2 = 1$, $Z \geq 1$

Gaussian curvature of comparative metrics of the second kind. Let the *comparative metric* of the second type be given: $(ds')^2 = (du)^2 + \frac{1}{\alpha^2}\operatorname{sh}^2(\alpha u)(dv)^2$, where $\{\alpha \neq 0, \alpha \neq \pm 1, u>0, -\infty < v < +\infty\}$.

The coefficients of this metric are the following: $E'=1>0$, $F'=0$, $G'=\frac{1}{\alpha^2}\operatorname{sh}^2(\alpha u)>0$.

The Gaussian curvature for this case is equal: $K' = -\alpha^2 < 0$. It follows from here that the radius of curvature R' of the second *comparative metric* of the second type is equal: $R' = \frac{1}{\sqrt{-K'}} = \frac{1}{\sqrt{\alpha^2}} = \frac{1}{|\alpha|} > 0$

$\Rightarrow R'^2 = \frac{1}{\alpha^2} > 0.$

Because $\{\alpha \neq 0, \alpha \neq \pm 1\}$, then from the equalities $K' = -\alpha^2$, $R' = \frac{1}{|\alpha|}$, $R'^2 = \frac{1}{\alpha^2}$ we get the following reletions: $0 > K' \neq -1$, $0 < R' \neq 1$, $0 < R'^2 \neq 1$.

The second *comparative metric* is realized under parameterization

$$X' = \frac{1}{\alpha}\operatorname{sh}(\alpha u)\cos(v),\ Y' = \frac{1}{\alpha}\operatorname{sh}(\alpha u)\sin(v'),\ Z' = \frac{1}{\alpha}\operatorname{ch}(\alpha u)$$

on the pseudo-sphere $M'^{2}: Z'^2 - X'^2 - Y'^2 = \frac{1}{\alpha^2} \neq 1$, $Z' \geq \frac{1}{|\alpha|} \neq 1$ in the three-dimensional Minkowski space (X, Y, Z), endowed with the Minkowski metric $(dl)^2 = (dZ)^2 - (dX)^2 - (dY)^2$.

On the pseudo-sphere $M'^{2}: Z'^2 - X'^2 - Y'^2 = \frac{1}{\alpha^2}$, $Z' \geq \frac{1}{|\alpha|}$ with the above parameterization of the *comparative metric*, we obtain the relationship:

$$-[(dZ')^2 - (dX')^2 - (dY')^2] = (ds')^2 = (du)^2 + \frac{1}{\alpha^2}\operatorname{sh}^2(\alpha u)(dv)^2 .$$

Thus, for $\rho = \left|\alpha^2 - 1\right| > 0$, the *comparative metric* of the second type has the following form: $(du)^2 + \frac{1}{\alpha^2}\text{sh}^2(\alpha u)(dv)^2$ $\{\alpha \neq 0, \alpha \neq \pm 1,\ u>0,\ -\infty < v < +\infty\}$ and has another geodesic curvature $K' = -\alpha^2$, than the geodesic curvature $K = -1$ of the *basic Lobachevsky metric* $(ds)^2 = (du)^2 + \text{sh}^2(u)(dv)^2$. Two different pseudo-spheres: $Z^2 - X^2 - Y^2 = 1$, $Z \geq 1$ (for the *basic Lobachevsky metric*) and $Z'^2 - X'^2 - Y'^2 = \frac{1}{\alpha^2}$, $Z' \geq \frac{1}{\alpha^2}$ (for the *comparative metric* of the second type) proved to be the carrier of these two metrics.

Comparison of metric properties for the metrics of the first type with $\{u>0,\ v \in (-\infty, +\infty)\}$. Let us show that with $\rho = \left|\alpha^2 - 1\right| > 0$, $\{\alpha \neq 0, \alpha \neq \pm 1\}$ the *basic Lobachevsky metric* $(ds)^2 = (du)^2 + \text{sh}^2(u)(dv)^2$ and the *comparative metric* of the first type $(ds')^2 = \alpha^2(du)^2 + \text{sh}^2(\alpha u)(dv)^2$ have opposite metric properties.

Nonisometry with $\rho = \left|\alpha^2 - 1\right| > 0$, $\{\alpha \neq 0, \alpha \neq \pm 1\}$ **for the metric of the first type.** According to the metric table, in order that the displaying $f: M^2 \mapsto M'^2$ would be *isometric* (preserved lengths), it is necessary and sufficient that the coefficients of metric forms coincide for parameterization of one and the same area of the plane of the parameters of these surfaces [23]. In our situation when comparing the metric forms $(ds)^2 = (du)^2 + \text{sh}^2(u)(dv)^2$ (the *Lobachevsky metric*) and $(ds')^2 = [\alpha^2(du)^2 + \text{sh}^2(\alpha u)(dv)^2]$ (the *comparative metric*) at the parameterization of the surfaces M^2 and M'^2 in one and the same area $\{0 < u < +\infty, -\infty < v < +\infty\}$ of the plane of parameters, the following equalities $E = E'$, $F = F'$, $G = G'$ were performed. Here the coefficients of the metric forms have the following forms: $E = 1$, $F = 0$, $G = \text{sh}^2(u) > 0$ and $E' = \alpha^2$, $F' = 0$, $G' = \text{sh}^2(\alpha u) > 0$ with additional requirements $\{\alpha \neq 0, \alpha \neq \pm 1\}$.

According to the table of comparison of the metric properties [24], here and in the future, in order to establish *isometry*, *conformity* and *equiarity*, we can directly use the comparison of the coefficients of the corresponding metrics on surfaces. Let us apply a general algorithm, consisting in the use of expansion in absolutely convergent Taylor series.

Non-isometry. Suppose there is *isometry*. Then we get the equalities:

$E = E' \Rightarrow 1 = \alpha^2$, $0 = 0$, $G = G' \Rightarrow \text{sh}^2(u) = \text{sh}^2(\alpha u)$. But for this case we get the equalities: $E = E' \Rightarrow 1 = \alpha^2 \Rightarrow a = \pm 1$ what contradicts to the condition $\rho > 0$, $\{\alpha \neq 0, \alpha \neq \pm 1\}$. Therefore, in this situation, we get the inequality: $E \neq E'$, that is, it is *nonisometry*.

We also show that under the condition $\{\alpha \neq 0, \alpha \neq \pm 1,\ u>0\}$ we get also that $G = \text{sh}^2(u) \neq G' = \text{sh}^2(\alpha u)$. Let's suppose the contrary, that is, that the following equality exists: $\text{sh}^2(u) = \text{sh}^2(\alpha u)$, where $\{\alpha \neq 0, \alpha \neq \pm 1,\ u>0\}$. Then, we get: $\text{sh}^2(u) - \text{sh}^2(\alpha u) = 0$. Let's decompose the function $P_1 = \text{sh}^2(u) - \text{sh}^2(\alpha u)$ in a Taylor series on the variable u with the center of decomposition $u_0 = 0$. Then, we get:

$$P_1 = (1-\alpha^2)u^2 + (\frac{1}{3}(1-\alpha^4)u^4 + \frac{2}{45}(1-\alpha^6)u^6 + \frac{1}{315}(1-\alpha^8)u^8 + \ldots = 0.$$

Because $\{\alpha \neq 0, \alpha \neq \pm 1$, $u>0\}$ and $P_1=0$, then we can divide this series by $(1-\alpha^2)u^2$. Then after all the cuts we get:

$$P_2 = \frac{P_1}{(1-a^2)u^2} = \frac{1}{3}(1+\alpha^2)u^2 + \frac{2}{45}(1+\alpha^2+\alpha^4)u^4 + \frac{1}{315}(1+\alpha^2+\alpha^4+\alpha^6)u^6 + \ldots = 0$$

All members of this series are positive and, moreover, $P_2>1$, what is contrary to $P_2=0$. But then under the conditions $\{\alpha \neq 0, \alpha \neq \pm 1$, $u>0\}$ we have *nonisometry* (not save lengths) between the *basic Lobachevsky metric* $(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2$ and the *comparative metric of the first type* $(ds')^2 = \alpha^2(du)^2 + \mathrm{sh}^2(\alpha u)(dv)$.

Nonconformity with $\rho = |\alpha^2 - 1| > 0$, $\{\alpha \neq 0, \alpha \neq \pm 1\}$ for the metric of the first type. The conformal displaying preserves the angles between curves at its intersection points (preservation of angles). Let's show that in the case of $\{\alpha \neq 0, \alpha \neq \pm 1\}$ for the case $\rho = |\alpha - 1| > 0$ there is *nonconformity*. Suppose the contrary, that is, that there is a conformal displaying $f\colon M^2 \mapsto M'^2$ Then, under the above conditions $\{\alpha \neq 0,\ \alpha \neq \pm 1\}$,$\{u>0\}$ there must be such a function

$\lambda_0 = \lambda_0\ (u,v) > 0$, so that $E = \lambda_0 E'$, $F = \lambda_0 F'$, $G = \lambda_0 G'$.

Because $E=1$, $F=0$, $G = \mathrm{sh}^2(u) > 0$ and $E' = \alpha^2$, $F'=0$, $G' = \mathrm{sh}^2(\alpha u) > 0$, that from the conditions $E = \lambda_0 E'$, $F = \lambda_0 F'$, $G = \lambda_0 G'$ we get the following equalities:

$$1 = \lambda_0 \alpha^2,\ \mathrm{sh}^2(u) = \lambda_0 \mathrm{sh}^2(\alpha u) \Rightarrow \lambda_0 = \frac{1}{\alpha^2},\ \mathrm{sh}^2(u) = \frac{1}{\alpha^2}\mathrm{sh}^2(\alpha u)$$
$$\Rightarrow \alpha^2\ \mathrm{sh}^2(u) = \mathrm{sh}^2(\alpha u) \Rightarrow P_1 = \alpha^2\,\mathrm{sh}^2(u) - \mathrm{sh}^2(\alpha u) = 0.$$

Let's decompose the function $P_1 = \alpha^2\,\mathrm{sh}^2(u) - \mathrm{sh}^2(\alpha u)$ in the Taylor series on the variable u with the center of the decomposition $u_0 = 0$. Then, we get:

$$P_1 = \frac{1}{3}(\alpha^2-\alpha^4)u^4 + \frac{2}{45}(\alpha^2-\alpha^6)u^6 + \frac{1}{315}(\alpha^2-\alpha^8)u^8 + \frac{2}{14175}(\alpha^2-\alpha^{10})u^{10} + \ldots = 0$$

Because $\{\alpha \neq 0, \alpha \neq \pm 1$, $u>0\}$, then $(\alpha^2-\alpha^4)u^4 = \alpha^2(1-\alpha^2)u^4 \neq 0$. Then, the function $P_2 = \frac{3}{(\alpha^2-\alpha^4)u^4} P_1$ is decomposed into the Taylor series as follows: $P_2 = 1 + \frac{2}{15}(1+a^2)u^2 + \frac{1}{105}(1+\alpha^2+\alpha^4)u^4 + \frac{2}{4725}(1+\alpha^2+\alpha^4+\alpha^6)u^6 + \ldots = 0$.

Because each member of this series is positive and, moreover, $P_2>1$, then we get a contradiction in the form: $0 = P_2 > 1$ what is impossible. Thus, *nonconformity* with $\rho > 0$ for the condition $\{\alpha \neq 0, \alpha \neq 1\}$ has been proved.

Aquirealirty for $\rho = \left|\alpha \ -1\right|$ >0, $\{\alpha \neq 0, \alpha \neq 1\}$ for the metrics of the first type. Aquireal displaying preserves the area of geometric figures. Let's show that in the case of $\{\alpha \neq 0, \alpha \neq 1\}$ for $\rho = \left|\alpha^2 - 1\right|$ there is *equiarity* of metrics. Suppose the contrary, that is, that there is an aquireal displaying $f: M^2 \mapsto M'^2$. Then under the above conditions $\{\alpha \neq 0,\ \alpha \neq \pm 1\}$,{$u$>0} the following equality will be performed: $EG-(F)^2 = E'G' - (F')^2$. The coefficients of the *basic Lobachevsky metric* $(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2$ have the following form:

E=1, F=0, G= $\mathrm{sh}^2(u) > 0$, but the coefficients of the *comparative metric* $(ds')^2 = \alpha^2 (du)^2 + \mathrm{sh}^2(\alpha u)(dv)^2$ have the form: $E' = \alpha^2 > 0$, $F'=0$, $G' = \mathrm{sh}^2(\alpha u) > 0$.

Therefore, for this situation, the equality $EG - (F)^2 = E'G' - (F')^2$ has the form: $\mathrm{sh}^2(u) - \alpha^2 \mathrm{sh}^2(\alpha u) = 0$. Let's decompose the function $P_1 = \mathrm{sh}^2(u) - \alpha^2 \mathrm{sh}^2(\alpha u)$ into Taylor series on the variable u with the center of decomposition $u_0 = 0$. Then we get:

$$P_1 = (1-\alpha^4)\, u^2 + \frac{1}{3}(1-\alpha^6)u^4 + \frac{2}{45}(1-\alpha^8)u^6 + \frac{1}{315}(1-\alpha^{10})u^8 + \ldots = 0.$$

Because $\{\alpha \neq 0, \alpha \neq \pm 1,\ u > 0\}$, then $(1-\alpha^4)\,u^2 \neq 0$. Then the function $P_2 = \frac{1}{(1-\alpha^4)u^2} P_1$ is decomposed into Taylor series as follows:

$$P_2 = 1 + \frac{1}{3}\left(\frac{1+\alpha^2+\alpha^4}{1+\alpha^2}\right)u^2 + \frac{2}{45}(1+\alpha^4)u^2 + \frac{1}{315}\left(\frac{1+\alpha^2+\alpha^4+\alpha^6+\alpha^8}{1+\alpha^2}\right)u^6 + \ldots = 0.$$

However, because every member of this series is positive and, moreover, $P_2 > 1$, then we get the following contradiction: $0 = P_2 > 1$, what is impossible. Thus, the *nonequiarity* with $\rho > 0$ for the condition $\{\alpha \neq 0, \alpha \neq \pm 1\}$ has been proved.

So, when $\{\alpha \neq 0, \alpha \neq \pm 1\}$, $\{0 < u < +\infty, -\infty < v < +\infty\}$ for the case $\rho = \left|\alpha^2 - 1\right| > 0$, for the comparison of the *basic Lobachevsky metric* $(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2$ to the metric $(ds')^2 = \alpha^2 (du)^2 + \mathrm{sh}^2(\alpha u)(dv)^2$, there is *nonisometry* (the lengths are not preserved), there is *nonconformity* (the angles are not preserved) and there is *nonequiarity* (the areas are not preserved).

The peculiarity of this result consists in the fact that in this case the Gaussian curvatures of the *basic Lobachevsky metric* $(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2$ and the *comparative metrics* of the type $(ds')^2 = \alpha^2 (du)^2 + \mathrm{sh}^2(\alpha u)(dv)^2$ for the conditions $\{\alpha \neq 0, \alpha \neq \pm 1\}$, $\{0 < u < +\infty, -\infty < v < +\infty\}$, are equal, that is, $K = K' = -1$.

Comparison of metric properties for metrics of the second type with

{u>0, $v \in (-\infty, +\infty)$}.

Let's show that with $\rho = \left|\alpha^2 - 1\right| > 0$, $\{\alpha \neq 0, \alpha \neq \pm 1\}$ the *basic Lobachevsky metric* $(ds^2) = (du)^2 + \text{sh}^2(u)(dv)^2$ and the *comparative metric* $(ds')^2 = (du)^2 + \frac{1}{\alpha^2}\,\text{sh}^2(\alpha u)(dv)^2$ have opposite metric properties

Nonisometry at $\rho = \left|\alpha^2 - 1\right| > 0$, $\{\alpha \neq 0, \alpha \neq \pm 1\}$ for the metrics of the second type.

According to the metric table, in order the *displaying* $f: M^2 \mapsto M'^2$ was *isometric,* it is necessary and sufficient, so that the coefficients of the metric forms coincide, when the parameterization in the same area of the plane of parameters of these surfaces was realized [24]. In our situation when comparing metrics $(ds)^2 = (du)^2 + \text{sh}^2(u)(dv)^2$ (the *basic Lobachevsky metric*) and $(ds')^2 = (du)^2 + \frac{1}{\alpha^2}\,\text{sh}^2(\alpha u)(dv)^2$ (the *comparative metric*) at parameterization of the surfaces M^2 and M'^2 in one and the same area $\{u > 0, -\infty < v < +\infty\}$ of parameters plane; this means that the following equalities are performed: $E = E'$, $F = F'$, $G = G'$. Here the coefficients of metric forms are the following:

$$E=1,\ F=0,\ G=\text{sh}^2(u)>0 \text{ и } E'=1,\ F'=0,\ G'=\frac{1}{\alpha^2}\text{sh}^2(\alpha u)>0$$

with additional requirements $\{\alpha \neq 0, \alpha \neq \pm 1\}$. Note that in this situation, the Gaussian curvature of the *basic Lobachevsky metric* is equal to $K = -1$, but the *comparative metric* is equal to $K' = -\alpha^2 < 0$. Because $\{\alpha \neq 0, \alpha \neq \pm 1\}$, then $K \neq K'$. Recall the Gaussian theorem [23] (a necessary condition for the constancy of the Gaussian curvature):

> "*If isometry with the displaying (the lengths are preserved), then the Gaussian curvature at the corresponding points is the same".*

However, these conditions are necessary, but not sufficient, that is, if the Gaussian curvature at the corresponding points is the same, then the *displayings*, a priori, can be *nonisometric*. Namely, for the metrics of the first type, when $\{\alpha \neq 0, \alpha \neq \pm 1\}$, the Gaussian curvatures were the same ($K = K' = -1$), but there was *nonisometry* (not preserve the lengths) and, moreover, there was also *nonconformal* (not preserve the the angles) and *aquireal* (not preserve the areas). If at the corresponding points the Gaussian curvatures do not coincide, then there is certainly *nonisometry*, because, for example, in this case $K = -1$, $K' = -\alpha^2$, $K \neq K'$, where $\{\alpha \neq 0, \alpha \neq \pm 1\}$. Therefore, in this situation, from the Gaussian theorem on *isometry* for the conditions $E=1$, $F=0$, $G=\text{sh}^2(u)>0$ и $E'=1$, $F'=0$, $G'=\frac{1}{\alpha^2}\text{sh}^2(\alpha u)$ we get, that because $E=E'=1$, $F=F'=0$, but we have *nonisometry*, then there follows the following result: $G=\text{sh}^2(u) \neq G'=\frac{1}{\alpha^2}\text{sh}^2(\alpha u)$.

Nonconformity with $\rho = \left|\alpha^2 - 1\right| > 0$, $\{\alpha \neq 0, \alpha \neq \pm 1\}$ for the metrics of the second type. Suppose the contrary, that is, there is the *conformal displaying* $f: M^2 \mapsto M'^2$. Then under the above conditions $\{\alpha \neq 0,\ \alpha \neq \pm 1, u > 0, -\infty < v < +\infty\}$, when we compare the *basic Lobachevsky metric* $(ds)^2 = (du)^2$

$+\mathrm{sh}^2(u)(dv)^2$ with any fixed *comparative metric* $(ds')^2 = (du)^2 + \frac{1}{\alpha^2}\,\mathrm{sh}^2(\alpha u)(dv)^2$ there must be a such function $\lambda_0 = \lambda_0(u,v) > 0$, so that $E = \lambda_0 E'$, $F = \lambda_0 F'$, $G = \lambda_0 G'$.

Because $E=1$, $F=0$, $G = \mathrm{sh}^2(u) > 0$ and $E'=1$, $F'=0$, $G' = \frac{1}{\alpha^2}\mathrm{sh}^2(\alpha u) > 0$, then, from the conditions $E = \lambda_0 E'$, $F = \lambda_0 F'$, $G = \lambda_0 G'$ we get the following equalities:

$1 = \lambda_0$, $\mathrm{sh}^2(u) = \lambda_0 \frac{1}{\alpha^2}\mathrm{sh}^2(\alpha u) \Rightarrow \mathrm{sh}^2(u) - \frac{1}{\alpha^2}\mathrm{sh}^2(\alpha u) = 0 \Rightarrow \alpha^2\,\mathrm{sh}^2(u) - \mathrm{sh}^2(\alpha u) = 0.$

Next, apply the same algorithm for the Taylor expansion of the function $P_1 = \alpha^2\,\mathrm{sh}^2(u) - \mathrm{sh}^2(\alpha u)$ and divide this series by the first coefficient; then, for this case we get that this series is *greater than zero*, but on the other hand, this series is *zero* what is impossible. Therefore, in this situation there is also *nonconformity*.

Nonaquirealirty with $\rho = \left|\alpha^2 - 1\right| > 0$, $\{\alpha \neq 0, \alpha \neq \pm 1\}$ **for the metrics of the second type.** The aquireal displaying preserves the areas of the corresponding geometric figures. Suppose the contrary, that is, that there is an *aquireal displaying* $f : M^2 \mapsto M'$. Then, under the above conditions $\{\alpha \neq 0, \alpha \neq \pm 1, u > 0, -\infty < v < +\infty\}$, according to [23] (a table of comparison of metric properties), for *aquirealirty* it is necessary and sufficient that when comparing the corresponding coefficients of metric forms satisfy to the equality: $EG - (F)^2 = E'G' - (F')^2$. In our situation $\{\alpha \neq 0, \alpha \neq \pm 1, u > 0, -\infty < v < +\infty\}$ by assuming *aquirealirty* between the *basic Lobachevsky metric* $(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2$ and the *comparative metric* $(ds')^2 = (du)^2 + \frac{1}{\alpha^2}\,\mathrm{sh}^2(\alpha u)(dv)^2$ it is necessary and sufficient so that the following equality is performed: $EG - (F)^2 = E'G' - (F')^2$. Here we have: $E=1$, $F=0$, $G = \mathrm{sh}^2(u) > 0$, $E'=1$, $F'=0$, $G' = \frac{1}{\alpha^2}\mathrm{sh}^2(\alpha u) > 0$. But then we get:

$$EG - (F)^2 = E'G' - (F')^2 \Rightarrow \mathrm{sh}^2(u) = \frac{1}{\alpha^2}\mathrm{sh}^2(\alpha u) \Rightarrow \alpha^2\,\mathrm{sh}^2(u) - \mathrm{sh}^2(\alpha u) = 0.$$

This situation $\alpha^2\,\mathrm{sh}^2(u) - \mathrm{sh}^2(\alpha u) = 0$ met, when proving *nonconformity* with $\rho = \left|\alpha^2 - 1\right| > 0$, $\{\alpha \neq 0, \alpha \neq \pm 1\}$ for the metrics of the first type. It has been shown that this situation is impossible. Therefore, we obtain that there follows from comparisons of the metrics

$$(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv) \text{ and } (ds')^2 = (du)^2 + \frac{1}{\alpha^2}\,\mathrm{sh}^2(\alpha u)(dv)^2$$

that the metrics are *nonaquireal* (the areas are not preserved) for the case $\{u > 0, v \in (-\infty, +\infty)\}$. Therefore, when comparing the second type of the *comparative metrics* to the *basic Lobachevsky metrix*, we get *nonisometry*, *nonconformity* and *nonaquireality.*

The third type of metrics comparison for the case $\{u > 0, v \in (-\infty, +\infty)\}$

$$(ds_1)^2 = \alpha^2 (du)^2 + \mathrm{sh}^2(\alpha u)(dv)^2 \text{ (Gaussian curvature } K_1 = -1),$$

$(ds_2)^2 = \beta^2 (du)^2 + \mathrm{sh}^2(\beta u)(dv)^2$ (Gaussian curvature $K_2 = -1$)

for the conditions $\{\alpha^2 \neq \beta^2, \alpha \neq 0, \alpha \neq \pm 1, \beta \neq 0, \beta \neq \pm 1, u>0, v \in (-\infty,+\infty)\}$.

Representation of the third type of metrics comparison in terms of hyperbolic Fibonacci λ-functions. Let's assume that

$\alpha = \ln(\Phi_\lambda) \Leftrightarrow \lambda = 2\,\mathrm{sh}(\alpha)$, $\beta = \ln(\Phi_\mu) \Leftrightarrow \mu = 2\,\mathrm{sh}(\beta)$ for the conditions $\alpha^2 \neq \beta^2$, $\alpha \neq 0, \alpha \neq \pm 1, \beta \neq 0, \beta \neq \pm 1, u>0, v \in (-\infty,+\infty)\} \Leftrightarrow$ $\{\ln^2(\Phi_\lambda) \neq \ln(\Phi_\mu), \lambda \neq 0, \lambda \neq \pm 2.3504, \mu \neq 0, \mu \neq \pm 2.3504, u>0, v \in (-\infty,+\infty)\}$.

Then, we get the two metrics, the first *comparative metric*

$$(ds_1)^2 = \ln^2(\Phi_\lambda)(du)^2 + \mathrm{sh}^2[u \bullet \ln(\Phi_\lambda)](dv)^2 = \ln^2(\Phi_\lambda)(du)^2 + \frac{4+\lambda^2}{4} sF_\lambda^2(u)(dv)^2$$

(Gaussian curvature $K_1 = -1$), and the second *comparative metric*

$$(ds_2)^2 = \ln^2(\Phi_\mu)(du)^2 + \mathrm{sh}^2[u \bullet \ln(\Phi_\mu)](dv)^2 = \ln^2(\Phi_\mu)(du)^2 + \frac{4+\mu^2}{4} sF_\mu^2(u)(dv)^2$$

(Gaussian curvature $K_2 = -1$).

The fours type of metrics comparison for the case $\{u>0, v \in (-\infty,+\infty)\}$

$(ds_1)^2 = (du)^2 + \frac{1}{\alpha^2}\mathrm{sh}^2(\alpha u)(dv)^2$ (Gaussian curvature $K_1 = -\alpha^2 < 0$)

$(ds_2)^2 = (du)^2 + \frac{1}{\beta^2}\mathrm{sh}^2(\beta u)(dv)^2$ (Gaussian curvature $K_2 = -\beta^2 < 0$)

for the conditions $\{\alpha^2 \neq \beta^2, \alpha \neq 0, \alpha \neq \pm 1, \beta \neq 0, \beta \neq \pm 1, u>0, v \in (-\infty,+\infty)\}$.

Representation of the fourth kind of comparison of metrics in terms of hyperbolic Fibonacci λ-functions. Let's assume that

$\alpha = \ln(\Phi_\lambda) \Leftrightarrow \lambda = 2\,\mathrm{sh}(\alpha)$, $\beta = \ln(\Phi_\mu) \Leftrightarrow \mu = 2\,\mathrm{sh}(\beta)$ for the conditions $\{\alpha^2 \neq \beta^2$, $\alpha \neq 0, \alpha \neq \pm 1, \beta \neq 0, \beta \neq \pm 1, u>0, v \in (-\infty,+\infty)\} \Leftrightarrow$ $\{\ln^2(\Phi_\lambda) \neq \ln(\Phi_\mu), \lambda \neq 0, \lambda \neq \pm 2.3504, \mu \neq 0, \mu \neq \pm 2.3504, u>0, v \in (-\infty,+\infty)\}$

Then, we get the two metrics, which are expressed through the Spinadel's *metallic proportions* Φ_λ and Φ_μ :

$$(ds_1)^2 = (du)^2 + \frac{1}{\ln^2(\Phi_\lambda)}\mathrm{sh}^2[u \bullet \ln(\Phi_\lambda)](dv)^2 = (du)^2 + \frac{4+\lambda^2}{\ln^2(\Phi_\lambda^2)} sF_\lambda^2(u)(dv)^2$$

(Gaussian curvature $K_1 = -\ln^2(\Phi_\lambda) < 0$), and $(ds_2)^2 = (du)^2 + \dfrac{1}{\ln^2(\Phi_\mu)} \operatorname{sh}^2 [u \bullet \ln(\Phi_\mu)]\,(dv)^2 = (du)^2 + \dfrac{4+\mu^2}{\ln^2(\Phi_\mu^2)}\, sF_\mu^2(u)\,(dv)^2$

(Gaussian curvature $K_2 = -\ln^2(\Phi_\mu) < 0$), where $K_2 \neq K_1$.

Both types of these comparisons for the *comparative metrics* themselves on the subject of *nonisometry*, *nonconformity* and *nonaquireality* are conducted in a similar way (with a slight modification) by using the general algorithm for decomposition into Taylor series. In these last two cases, the function, taken as the distance between the comparative metrics is $\rho(\alpha,\beta) = \left|\alpha^2 - \beta^2\right|$.

In terms of the Mathematics of Harmony [1] after the replacements

$\alpha =$ ln (Φ_λ), $\beta =$ ln (Φ_μ), the distance between the *comparative metrics* looks like $\overline{\rho}(\lambda,\mu) = \left|\ln^2(\Phi_\lambda) - \ln^2(\Phi_\mu)\right|$. For the case $\lambda = \pm\mu$ we get:
$\overline{\rho}(\lambda,\mu) = \left|\ln^2(\Phi_\lambda) - \ln^2(\Phi_\mu)\right| = 0$.

7 New Challenge for Theoretical Natural Sciences: Insolvability of the 4-th Hilbert Problem

Theorem 2 (Complete solution of the Fourth Hilbert Problem for hyperbolic geometries). *The Fourth Hilbert's Problem is insoluble for hyperbolic geometries.*

Proof:

Note that taking into consideration the above arguments, the authors' solution of the 4-th Hilbert Problem, which is described in [24], can be considered not only as *variant of the particular solution* of this problem (the first approach), but also as the *complete solution* of the 4-th Hilbert Problem (the second approach), unlike of the particular solutions of Hamel, Pogorelov and others researchers. Namely, the authors of this article proved the existence of an infinite number of new hyperbolic geometries, arbitrarily near to Lobachevsky's geometry, but having other metric properties in comparison with Lobachevsky's geometry (*nonisometry, nonconformity, nonaquireality*). The authors used one and the same general algorithm for comparison these metrics to find their metric properties. This algorithm allows the comparison of the comparative metrics to the Lobachevsky metric and the comparative metrics among themselves; this algorithm is based on the decomposition of the remainders between metrics into absolutely convergent Taylor series. If such remainders after division on the first term have constant signs, then the corresponding metric properties do not match. If in such series, after division on the first term, we get alternating variables, then by using this general algorithm it is impossible to establish directly, whether the corresponding metric properties of the compared metrics are coincident or differ. To do this, we always need to search for specific ways and algorithms.

But because the set of geometries, near to Lobachevsky's geometry, is infinite, we certainly come to the conclusion, that for an infinite set of geometries, near to Lobachevsky's geometry, it is impossible to find one and the same general algorithm, which makes possible for any metrics from this infinite set to define to have or don't have other metric properties than the metric properties of Lobachevsky's geometry or to draw a similar conclusion after comparison of the metrics to each other. This set of metrics can be used for comparison of the metric properties of both metrics with the same Gaussian curvature and metrics with different Gaussian curvatures.

In particular, this set contains all metrics with different Gaussian curvatures. When comparing any two such metrics of the form $(ds)^2 = E(du)^2 + 2Fdudv + G(dv)^2$ (there are an infinite set of such metrics), if we apply a general algorithm for comparison of such metrics and we get the sign-alternating Taylor power series and therefore this general comparison method does not fit.

On the other hand, when we compare such metrics, the Gauss theorem is partially (but incomplete) applicable: if two such metrics have different Gaussian curvatures, then there is *nonisometry*. But it does not at all follow from this that there is no possible *conformity* and *aquireality*. In order to establish the presence or absence of *conformity* and *aquireality*, it is necessary in this situation to search for a specific method every time.

This, in any sense, is analogous to the fact that in a binary graph the set of all vertices is countable, and the set of all paths is countable (the power of the continuum).

Such an approach is in some sense similar to the proof of the insolvability of the 10-th Hilbert Problem (is there a universal algorithm for solving arbitrary Diophantine equations), made by the Russian mathematician Yuri Matiyasevich in 1970 [25,26]. Recall that the basic idea of the proof of the insolvability of the 10-th Hilbert Problem consisted in the fact that since the set of all Diophantine equations is uncountable and then, according to the main Matiyasevich theorem, *"the same general method (algorithm) is impossible, which allows for any Diophantine equations determining, whether they have a solution in integers or not. "*

Comparing the complete solution of the 4th problem of Hilbert, obtained in this article, with the solution of the 10-th Hilbert Problem, obtained by Yuri Matiyasevich in 1970, it is appropriate to draw attention to the following interesting fact. The ancient Golden Section, described in Euclid's Elements (Proposition II.11) and the following from it Mathematics of Harmony [9], as a new direction in geometry, are the main mathematical apparatus for the completed solution of the 4-th Hilbert Problem. By the way, this solution is reminiscent of the insolvability of the 10-th Hilbert Problem for Diophantine equations in integers. This outstanding mathematical result was obtained by Yuri Matiyasevich in 1970, by using **Fibonacci numbers**, introduced in 1202 by the famous Italian mathematician Leonardo from Pisa (by the nickname Fibonacci), and the new theorems in Fibonacci numbers theory, proved by the outstanding Russian mathematician Nikolay Vorobyev and described by him in the third edition of his book "Fibonacci numbers" [27]. But, as we know, thet Fibonacci numbers 1,1,2,3,5,8,13,…, $F_{n-1}, F_n, \ldots$ are a discrete analog of the Golden Proportion, because the ratio of neighbouring Fibonacci numbers $\frac{F_n}{F_{n-1}}$ in the limit tends to the Golden Proportion, that is,

$$\boxed{\lim_{n\to\infty}\frac{F_n}{F_{n-1}} = \frac{1+\sqrt{5}}{2}}$$

This famous formula was well known to Johannes Kepler and therefore in modern mathematical literature it is sometimes called the *Kepler formula*.

But after all, the Fibonacci numbers and the Golden Proportion are the expression of the mathematical harmony of Nature, in particular, the widely known botanical phenomenon of *phyllotaxis* [28]. Although the botanical phenomenon of phyllotaxis has been known since immemorial times, the *Fibonacci spirals*, that occur on the surface of many botanical objects (*pine cone, pineapple, cactus, sunflower head*) still remain the "mystery" of this unique botanical phenomenon.

That is why, the authors recommend to the readers to pay attention to the article [29], in which the particular solution of the 4-th Hilbert Problem, based on the Mathematics of Harmony [9], was called the MILLENIUM PROBLEM in geometry.

8 Conclusions

The particular solution of the 4-th Hilbert Problem is obtained; it is based on the hyperbolic Fibonacci λ-functions. The originality of this solution consists in the following:

1). This particular solution is based on the *Lobachevsky metric*, whose Gaussian curvature is equal to $K=-1$; this *Lobachevsky metric* is isometric to the *classical Lobachevsky metric* with the Gaussian curvature $K=-1$.

2). Two types of infinite set of the *comparative metrics*, based on hyperbolic Fibonacci λ-functions, are considered. These metrics can be arbitrarily near to the *basic Lobachevsky metric* and in the limit they coincide with the *basic Lobachevsky metric*.

The first type of all these *comparative metrics* has Gaussian curvature $K=-1$, the same with the *basic Lobachevsky metric*. However, all these comparative metrics with respect to the *basic Lobachevsky metric* are *nonisometric* (do not preserve lengths), *nonconformal* (do not preserve angles), *nonaquireal* (do not preserve areas). Moreover, these comparative metrics themselves are also *nonisometric*, *nonconformal* and *nonaquireal*. This particular solution is based on the *Lobachevsky metric*, whose Gaussian curvature is equal to $K=-1$; this *Lobachevsky metric* is isometric to the *classical Lobachevsky metric* with the Gaussian curvature $K=-1$.

Thus, the important conclusion of this study is proving the existence of an infinite set of new geometries, arbitrarily near to Lobachevsky's geometry and having the same with Lobachevsky's geometry negative Gaussian curvature $K=-1$.

The second type of all these *comparative metrics* has negative Gaussian curvatures $K(\lambda)=-\ln^2(\Phi_\lambda)<0)$, which differ from the Gaussian curvature $K=-1$ of the *basic Lobachevsky metric*. In relation to the *basic Lobachevsky metric*, all these *comparative metrics* are *nonisometric* (do not preserve lengths), *nonconformal* (do not preserve angles), *nonaquireal* (do not preserve areas).

Moreover, these *comparative metrics* themselves are also *nonisometric*, *nonconformal* and *nonaquireal*. In process of study, the authors of this article found metrics that clarify the Gauss theorem about the intrinsic geometry of surfaces, that for *displaying*, that preserve length (*isometry*), the Gaussian curvature remains the same.

It follows from the Gauss theorem that if two metrics, when compared, have different Gaussian curvatures, then, they are *nonisometric*. It follows from our study (for some particular situations) a revision of the Gauss theorem, that for any pairs of metrics, presented in the second type of specific metrics, not only between the *comparative metrics* and the *basic Lobachevsky's metric*, but also between specific metrics there exist not only *nonisometry* (according to the corollary to the Gauss theorem), but also *nonconformity* and *nonaquireality*.

Thus, it is proved the existence of an infinite number of new geometries (*nonisometric*, *nonconformal*, *nonaquireal* each other) with different negative Gaussian curvature; these geometries are arbitrarily near to the Lobachevsky hyperbolic geometry, they have other negative Gaussian curvature and are in comparison to Lobchevsky geometry such properties as *nonisometry, nonconformity* and *nonaquireality*

3) But the main result of this article is obtaining the complete solution of the 4-th Hilbert Problem for hyperbolic geometries; the essence of this result is the following:

The 4-th Hilbert Problem is insoluble for hyperbolic geometries.

4) In the book "The «Golden» Non-Euclidean Geometry. Hilbert's Fourth Problem, «Golden» Dynamical Systems, and the Fine-Structure Constant" [14] the authors investigated particular solutions to the 4-th Hilbert Problem not only for the case of *Lobachevsky hyperbolic geometry*, but also for the wider class of non-Euclidean geometries (*geometry of Riemann* (elliptic geometry), *non-Archimedean geometry*, and *Minkowski geometry*). Developing the approach, outlined in this article, in connection with the above classes of non-Euclidean geometry, the authors came to the hypothesis that the fundamental mathematical result, proved in this article (insolubility of the 4th Hilbert problem for hyperbolic geometries), possibly is valid for all types of non-Euclidean geometries what however, requires strict proof.

Competing Interests

Authors have declared that no competing interests exist.

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