Asian Journal of Advanced Research and Reports

10(4): 21-29, 2020; Article no.AJARR.55560
ISSN: 2582-3248

The Formation of Implicit Second Order Backward Difference Adam's Formulae for Solving Stiff Systems of First Order Initial Value Problems of Ordinary Differential Equations

Sabo J.[1*], Kyagya T. Y.[2] and Ayinde A. M.[3]

[1]Department of Mathematics, Adamawa State University, Mubi, Nigeria.
[2]Department of Mathematics and Statistics, Federal University, Wukari, Nigeria.
[3]Department of Mathematics, School of Physical Sciences, Modibbo Adama University of Technology, Yola, Nigeria.

Authors' contributions

This work was carried out in collaboration among all authors. Author JS developed the method. Author TYK analyzed the basic properties of the method. Author AMA test the method on some highly stiff ordinary differential equations. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/AJARR/2020/ v10i430249
Editor(s):
(1) Dr. Rachid Masrour, University of Cadi Ayyad, Morocco.
Reviewers:
(1) Adel H. Phillips, Ain-Shams University, Egypt.
(2) Shaibu Osman, University of Health and Allied Sciences, Ghana.
Complete Peer review History: http://www.sdiarticle4.com/review-history/55560

Original Research Article

Received 08 February 2020
Accepted 14 March 2020
Published 30 May 2020

ABSTRACT

The formation of implicit second order backward difference Adam's formulae for solving stiff systems of ODEs was study in this paper. We used interpolation and collocation in deriving backward differentiae Adam's formulae. The basic properties of our method was analyzed, and it was found to be consistent, zero-stability and convergent, we further plotted the region of absolute stability and it was shown to be A-stable. Numerical evidences shows that the multistep method develop is very effective method for in handling linear ODEs either initial value problems or boundary value problems.

Keywords: Implicit second order; backward difference; Adam's formulae and ODEs.

**Corresponding author: Email: sabojohn630@yahoo.com;*

1. INTRODUCTION

Most of these improvements in the class of linear multistep methods have been based on backward differentiation formula (BDF), because of its special properties. Among the first modifications introduced by different authors was the Extended Backward Differentiation formulas (EBDFs), introduced in 1980 by Cash, in which one-super future point technique was applied. Cash, [1], proposed Second derivative extended backward differentiation formulas for the numerical integration of stiff systems. As opposed to one-step methods, which only utilize one previous value of the numerical solution to approximate the subsequent value, multistep methods approximate numerical values of the solution by referring to more than one previous value. Accordingly, multistep methods may often achieve greater accuracy than one-step methods that use the same number of function evaluations, since they utilize more information about the known portion of the solution than one-step methods do.

Meanwhile, some scholars such as Meyer, Berghe and Vanthournout, [2], developed the Modified backward differentiation methods of Adams type based on exponential interpolation. A study of generalized Adams-Moulton method for the satellite orbit determination problem was study by Hong and Hahmwood, [3]. Skwame, Sabo, Kyagya and Bakari, [4], has developed a class of two-step second derivative Adam Moulton method with two off-step points for solving second order stiff ordinary differential equations. And, a sixth order implicit hybrid backward differentiation formulae (HBDF) for block solution of ordinary differential equations was study by Muhammad and Yahaya, [5]. Again, the study of second derivative hybrid block backward differentiation formula for numerical solution of stiff systems was carried out by Skwame, Kumlengand Bakari, [6]. Stuart and Humphries, [7], realized that there was an important class of ordinary differential equations (ODEs), which have become known as stiff equations, which presented a severe challenge to numerical methods that existed at that time. Since then an enormous amount of effort has gone into the analysis of stiff problems and, as a result, a great many numerical methods have been proposed for their solution. Their work motivated us to propose implicit second order backward difference Adam's formulae for solving stiff systems of ODEs of the form,

$$\left.\begin{aligned} y_1^{'} &= f_1(x, y_1, y_2, \cdots, y_n) \\ y_2^{'} &= f_2(x, y_1, y_2, \cdots, y_n) \\ &\vdots \\ y_n^{'} &= f_n(x, y_1, y_2, \cdots, y_n) \end{aligned}\right\} \tag{1.1}$$

subject to the initial conditions

$$\left.\begin{aligned} y_1(x_0) &= y_1^0 \\ y_2(x_0) &= y_2^0 \\ &\vdots \\ y_n(x_0) &= y_n^0 \end{aligned}\right\} \tag{1.2}$$

In order to achieve the aim and objectives of this paper, we shall interpolate, collocate and evaluate a power series approximate solution at some chosen grid and off-grid points via Cash, [8].

2. DERIVATION OF BACKWARD DIFFERENCE ADAMS-MOULTON FORMULAE (BDAMF)

The difference between Adams-Moulton and Adams-Bashforth methods is that Adams-Moulton methods use an interpolating polynomial of degree $\leq k$ rather than $\leq k-1$, and it includes f at the unknown value t_n as well, Moulton, [9]. A special category of multistep methods are the linear multi-step methods, where the numerical solution to the ODE (1.1) at a specific location is expressed as a linear combination of the numerical solution's values and the function's values at previous Points Cash, [1]. For the standard system of (1.1), a linear multistep method with $k-step$ would have the form:

$$\sum_{j=0}^{k} \alpha_{-j} y_{n-j} = h\sum_{j=0}^{k} \beta_{-j} f_{n-j} + h^2 \sum_{j=0}^{k} \delta_{-j} g_{n-j} \tag{2.1}$$

where α_j, β_j are constants, y_n is the numerical solution at $t = t_n \; and f_n = f(t_n, y_n)$.

In contrast to the linear multistep schemes in the Adams Family, who are derived by integrating an interpolating polynomial $\phi(t)$ that approximates f , the BDAMF are derived by differentiating an interpolating polynomial ϕ(t) that approximates y (one such that $\phi(t_{n-i}) = y(t_{n-i})\ for\ i = 0, 1, 2, \cdots, k$), and setting the derivative at t_n to be equal to $f(t_n, y_n)$, Hairer and Wanner, [10].

For example, the one-step BDF method is derived as follows. We first construct the interpolating polynomial $\varphi(t)$ that approximates y , with $\varphi(t_{n-i}) = y(t_{n-i})\ for\ i = 0, 1.$

$$y(t) \cong \varphi(t) = y(t_n) + (t - t_n)\frac{y(t_n) - y(t_{n-1})}{t_n - t_{n-1}} \tag{2.2}$$

upon differentiation (2.2), we get:

$$y'(t) = f(t, y) \approx \varphi'(t) = \frac{y(t_n) - y(t_{n-1})}{t_n - t_{n-1}} \tag{2.3}$$

we can then use the approximation in (2.2) as inspiration to construct our $1 - step$ BDF method, by setting $\varphi'(t_n) = f(t_n, y_n)$:

$$\left.\begin{aligned}\frac{y(t_n) - y(t_{n-1})}{h} = f(t_n, y_n)\end{aligned}\right\} \tag{2.4}$$

Similarly, we can construct a $k - step$ Backward Difference Adams Moulton's Formulae (BDAMF) by generating the $k - \deg ree$ interpolating polynomial Bashford and Adams, [11]:

$$y(t) \approx \varphi(t) = y_n + \frac{1}{h}(t - t_n)\nabla y_n + \frac{1}{2h^2}(t - t_n)(t - t_{n-1})\nabla^2 y_n + \cdots + \frac{1}{h^k k!}(t - t_n)\cdots(t - t_{n-k+1})\nabla^k \tag{2.5}$$

where ∇^i is the backward difference operator:

$$\nabla^0 y_n = y_n \tag{2.6}$$

$$\nabla^i y_n = \nabla^{i-1} y_n - \nabla^{i-1} y_{n-1} \tag{2.7}$$

then upon differentiating, and setting $\varphi'(t_n) = f(t_n, y_n)$, we get

$$\sum_{i=1}^{k} \frac{1}{i}\nabla^i y_n = hf(t_n, y_n) \tag{2.8}$$

which can be transformed to match the general expression of (2.1), with $\beta_j = 0\ for\ j > 0$ (note that this makes them implicit schemes):

$$y_n = \sum_{j=0}^{k} \alpha_j y_{n-j} + h \sum_{j=0}^{k} \beta_j f_{n-j} + h^2 \sum_{j=0}^{k} \delta_j g_{n-j} \tag{2.9}$$

from (2.9), we obtain the general form of the a $k-step$ Backward Difference Adams Moulton's formulae (BDAMF) as

$$y_n - \sum_{j=0}^{k} \alpha_{-j} y_{n-j} = h \sum_{j=0}^{k} \beta_{-j} f_{n-j} + h^2 \sum_{j=0}^{k} \delta_{-j} g_{n-j} \tag{2.10}$$

we obtain the continuous scheme for the single-step BDAMF by evaluating $k = -1, \alpha_{-j}, y_{n-j}, j = 0 \; and \; \beta_j, \delta_j, f_{n-j}, g_{n-j}, j = 0, -\frac{1}{4}, -\frac{1}{2}, -\frac{3}{4}, -1$ as follows

$$\alpha_0 = 0$$

$$\beta_0 = th - \frac{1}{17010} t^3 h \begin{pmatrix} 916650 + 6348825t + 21637224t^2 + 43701000t^3 + 54662400t^4 + 41630400t^5 \\ + 17704960t^6 + 3225600t^7 \end{pmatrix}$$

$$\beta_{-\frac{1}{4}} = -\frac{512}{8505} t^3 h \begin{pmatrix} 945 + 13230t + 66591t^2 + 172515t^3 + 256140t^4 + 220815t^5 \\ + 103040t^6 + 20160t^7 \end{pmatrix}$$

$$\beta_{-\frac{1}{2}} = \frac{8}{315} t^3 h \begin{pmatrix} 1890 + 17955t + 69678t^2 + 137760t^3 + 146880t^4 + 80640t^5 \\ + 17920t^6 \end{pmatrix}$$

$$\beta_{-\frac{3}{4}} = \frac{512}{8505} t^3 h \begin{pmatrix} 945 + 10710t + 52479t^2 + 140595t^3 + 220140t^4 + 200655t^5 \\ + 98560t^6 + 20160t^7 \end{pmatrix}$$

$$\beta_{-1} = \frac{1}{17010} t^3 h \begin{pmatrix} 100170 + 1172745t + 5987016t^2 + 16874760t^3 + 28074240t^4 + 27437760t^5 \\ + 14551040t^6 + 3225600t^7 \end{pmatrix}$$

$$\gamma_0 = \frac{1}{11340} t^2 h^2 \begin{pmatrix} 5670 + 63000t + 329175t^2 + 1002960t^3 + 1909320t^4 + 2304000t^5 \\ + 1713600t^6 + 716800t^7 + 129024t^8 \end{pmatrix}$$

$$\gamma_{-\frac{1}{4}} = \frac{32}{2835} t^3 h^2 \begin{pmatrix} 1890 + 17955t + 74214t^2 + 170520t^3 + 233640t^4 + 190260t^5 \\ + 85120t^6 + 16128t^7 \end{pmatrix}$$

$$\gamma_{-\frac{1}{2}} = \frac{8}{5} t^3 h^2 \left(256t^4 + 512t^3 + 376t^2 + 120t + 15\right)(t+1)^3$$

$$\gamma_{-\frac{3}{4}} = \frac{32}{2835} t^3 h^2 \begin{pmatrix} 630 + 7245t + 36162t^2 + 99120t^3 + 159480t^4 + 149940t^5 \\ + 76160t^6 + 16128t^7 \end{pmatrix}$$

$$\gamma_{-1} = \frac{1}{11340} t^3 h^2 \begin{pmatrix} 3780 + 44415t + 227808t^2 + 645960t^3 + 1082880t^4 + 1068480t^5 \\ + 573440t^6 + 129024t^7 \end{pmatrix}$$

On evaluating the continuous scheme at the same points, yields the hybrid block backward difference Adams Moulton's formula as follows.

$$y_n = y_{n-\frac{1}{4}} + \frac{1}{17418240}h\left(1539551f_n + 1429936f_{n-\frac{1}{4}} + 711936f_{n-\frac{1}{2}} + 613456f_{n-\frac{3}{4}} + 59681f_{n-1}\right)$$
$$-\frac{1}{11612160}h^2\left(26051g_n - 249656g_{n-\frac{1}{4}} - 183708g_{n-\frac{1}{2}} - 49720g_{n-\frac{3}{4}} - 2237g_{n-1}\right)$$

$$y_{n-\frac{1}{2}} - y_{n-\frac{1}{4}} = -\frac{1}{17418240}h\left(26081f_n + 1957456f_{n-\frac{1}{4}} + 2163456f_{n-\frac{1}{2}} + 193456f_{n-\frac{3}{4}} + 14111f_{n-1}\right)$$
$$+\frac{1}{11612160}h^2\left(893g_n + 55096g_{n-\frac{1}{4}} - 106596g_{n-\frac{1}{2}} - 13768g_{n-\frac{3}{4}} - 515g_{n-1}\right)$$

$$y_{n-\frac{3}{4}} = y_{n-\frac{1}{4}} - \frac{1}{68040}h\left(157f_n + 8402f_{n-\frac{1}{4}} + 16902f_{n-\frac{1}{2}} + 8402f_{n-\frac{3}{4}} + 157f_{n-1}\right)$$
$$+\frac{1}{90720}h^2\left(11g_n + 538g_{n-\frac{1}{4}} - 538g_{n-\frac{3}{4}} - 11g_{n-1}\right)$$

$$y_{n-1} = y_{n-\frac{1}{4}} - \frac{3}{71680}h\left(137\,f_n + 3792\,f_{n-\frac{1}{4}} + 6912\,f_{n-\frac{1}{2}} + 4912\,f_{n-\frac{3}{4}} + 2167\,f_{n-1}\right)$$
$$+\frac{3}{143360}h^2\left(15\,g_n + 488\,g_{n-\frac{1}{4}} + 756\,g_{n-\frac{1}{2}} + 744\,g_{n-\frac{3}{4}} - 113\,g_{n-1}\right)$$

3. ANALYSIS OF BACKWARD DIFFERENCE ADAMS MOULTON FORMULAE

In this section, the basic properties of backward difference Adams Moulton's formulae which include order, error constant, consistency, zero-stability, convergence and the stability region shall be analyzed.

3.1 Order and Error Constants of the Method

(Definition 3.1): Following Lambert, [12], the linear difference operator ℓ associated with the LMM (2.10) is defined by

$$\ell[y(x),h] = \sum_{j=0}^{k}\left[(a_j y(x+jh) + h\beta_j y'(x+jh) + h^2\beta_j y''(x+jh)\right] \tag{3.1}$$

where $y(x)$ is an arbitrary test function and it is continuously differentiable on $[a,b]$. Expanding $y(x+jh)\ and\ \ y'(x+jh)$ as Taylor series about x, and collecting common terms yields

$$\ell[y(x);h] = c_0 y(x) + c_1 h y'(x) + \cdots + c_q h^q y^q(x) + \cdots \tag{3.2}$$

where the constant $C_q, q = 0,1,\cdots$ coefficients are given as follows

$$c_0 = \alpha_0 + \alpha_1 + \quad \cdots + \alpha_k$$
$$c_1 = \alpha_1 + 2\alpha_2 + \quad \cdots + k\alpha_k - (\beta_0 + \beta_1 \cdots \beta_k)$$
$$\vdots$$
$$c_q = \tfrac{1}{q!}(\alpha_1 + 2^q\alpha_2 + \cdots + k^q\alpha_k) - \tfrac{1}{(q-1)!}(\beta_1 + 2^{(q-1)}\beta_2 + \cdots + k^{(q-1)}\beta_k), \quad q = 2,3,\cdots$$

According to Lambert, [12], the method (2.10) has order p if

$$c_0 = c_1 = \cdots = c_p = 0 \; and \; c_{p+1} \neq 0$$

of (3.2), then

Therefore the backward difference Adams Moulton's method (2.10) is of uniform order nine, where $p = 9$ is the order of the method and c_{p+2} is the error constant, given by

$$C_{p+2} = \left[4.1791\times10^{-13}, -6.7501\times10^{-14}, -1.3501\times10^{-13}, -5.5292\times10^{-13}\right]$$

3.2 Consistency

Lambert, [13], explained that consistency controls the magnitude of the local truncation error while zero stability controls the manner in which the error is propagated at each step of the calculation.

(Definition 3.2): The LMM (2.10) is said to be consistent if its order $P \geq 1$, It also follows from (3.2) that the LMM (2.10) is consistent if and only

$$\sum_{j=0}^{k} \alpha_j = 0$$
$$\sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \alpha_j \qquad (3.3)$$

It also follows from (3.3) that the LMM (2.10) is consistent if and only if

$$\left.\begin{aligned} \rho(1) &= 0 \\ \rho'(1) &= \sigma'(1) \end{aligned}\right\} \qquad (3.4)$$

According to (definition 3.1), the hybrid block (2.10) is consistent.

3.3 Zero Stability

(Definition 3.3): The LMM (2.10) said to be zero stable if the first characteristic polynomial $\pi(r)$ having roots such that $|r_z| \leq 1 \quad and \quad if \quad |r_z| = 1$, then the multiplicity of r_z must not be greater than two, Dahlquist, [14].

In order to find the zero-stability of hybrid block method (2.10), we only consider the first characteristic polynomial of the method according to Definition [3.3] as follows,

$$\Pi(r) = \left| r\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| = r^3(z-1)$$

which implies $r = 0, 0, 0, 1$. Hence the method is zero-stable since $|r_z| \leq 1$.

3.4 Convergence

Convergence is an essential property that every acceptable linear multistep method (LMM) must possess. According to Dahlquist, [14], consistency and zero stability are the necessary conditions for the convergence of any numerical method.

Theorem (3.1): The consistency and zero stability are sufficient condition for linear multistep method to be convergent. Since the hybrid block method (2.8) is consistent and zero stable, it implies that the method is convergent for all point Lambert, [13].

3.5 Region of Absolute Stability of the Block Method

The absolute stability region consists is the set of points in the complex plane outside the enclosed figure Following Lambert, [13] and Dahlquist, [14]. The absolute stability region of backward difference Adams Moulton's method (2.10) is obtained using the formula:

$$Aw - E_0 - E_1 - h^2 d - h^2 bw \qquad (3.5)$$

Simplify (2.11), yield the stability polynomial and the stability polynomial is then substituted into Matlab software program, Dahlquist, [14], and the region of absolute stability is shown below.

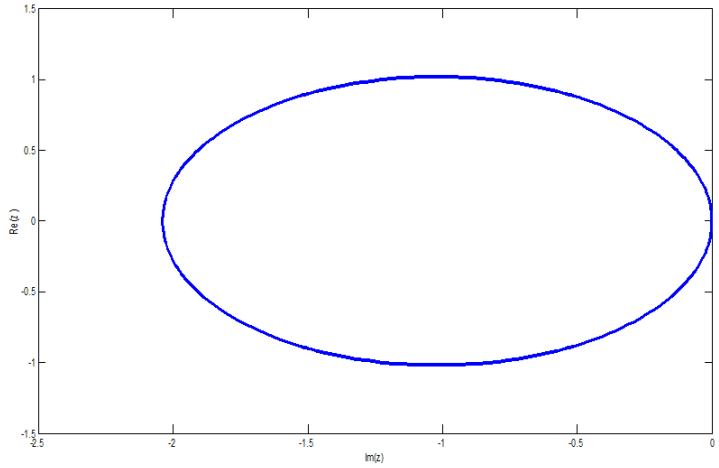

Fig. 1. Absolute Stability Region of the backward difference Adams method

4. NUMERICAL RESULTS

To illustrate the performance of our proposed BDAMF on three of stiff systems IVPs, we will compare our performance with the existing methods. The problems considered are the one solved by Skwame, Kumleng and Bakari, [6].

System 4.1

Consider the stiffly system,

$$y_1' = 198\ y_1 + 199\ y_2 \quad y_1(0) = 1$$
$$y_2' = -398\ y_1 - 399\ y_2 \quad y_2(0) = -1, \quad h = 0.1$$

With Exact Solution

$$y_1(x) = e^{-x}$$
$$y_2(x) = -e^{-x}$$
$$x \in [0,1]$$

Source, Skwame, Kumleng and Bakari, [6].

Table 1. Comparison of result of the new method with that of Skwame, Kumleng and Bakari, [6]

x	***Absolute errors in* Skwame, Kumleng and Bakari, [6]**				***Absolute error in New method***	
	$K = 2$ *and* $p = 6$		$K = 3$ *and* $p = 7$		$K = 2$ *and* $p = 10$	
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$
0.1	3.61×10^{-7}	3.60×10^{-7}	2.60×10^{-6}	2.60×10^{-6}	2.02×10^{-8}	2.14×10^{-8}
0.2	3.21×10^{-7}	3.30×10^{-7}	2.42×10^{-6}	2.42×10^{-6}	3.53×10^{-8}	3.65×10^{-8}
0.3	6.28×10^{-7}	3.27×10^{-7}	2.18×10^{-6}	2.18×10^{-6}	4.72×10^{-8}	4.84×10^{-8}
0.4	5.65×10^{-7}	5.65×10^{-7}	3.90×10^{-6}	3.90×10^{-6}	5.68×10^{-8}	5.76×10^{-8}
0.5	6.69×10^{-7}	6.68×10^{-7}	3.58×10^{-6}	3.58×10^{-6}	6.38×10^{-8}	6.47×10^{-8}
0.6	6.03×10^{-7}	6.02×10^{-7}	3.23×10^{-6}	3.23×10^{-6}	6.88×10^{-8}	7.96×10^{-8}
0.7	5.92×10^{-7}	5.92×10^{-7}	4.35×10^{-6}	4.35×10^{-6}	7.23×10^{-8}	7.29×10^{-8}
0.8	5.36×10^{-7}	5.37×10^{-7}	3.97×10^{-6}	3.97×10^{-6}	7.47×10^{-8}	7.53×10^{-8}
0.9	7.38×10^{-7}	7.38×10^{-7}	3.59×10^{-6}	3.59×10^{-6}	7.56×10^{-8}	7.62×10^{-8}
1.0	6.70×10^{-7}	6.70×10^{-7}	4.31×10^{-6}	4.30×10^{-6}	7.62×10^{-8}	7.69×10^{-8}

System 4.2

Consider the stiffly equation,

$$y_1' = -y_1; \quad y_1(0) = 1$$
$$y_2' = -2000 y_2; \quad y_2(0) = 1 \qquad h = 0.1, \quad 0 \le x \le 1$$

$y_1(x) = e^{-x}$

$y_2(x) = e^{-2000x}$

Table 2. Comparison of absolute error

X	Absolute errors	
	$y_1(x)$	$y_2(x)$
0.1	6.00×10^{-10}	9.52×10^{-1}
0.2	9.00×10^{-10}	9.10×10^{-1}
0.3	1.10×10^{-9}	8.62×10^{-1}
0.4	1.40×10^{-9}	8.20×10^{-1}
0.5	1.50×10^{-9}	7.80×10^{-1}
0.6	1.70×10^{-9}	7.42×10^{-1}
0.7	1.80×10^{-9}	7.10×10^{-1}
0.8	1.80×10^{-9}	6.72×10^{-1}
0.9	1.80×10^{-9}	6.39×10^{-1}
1.0	1.80×10^{-9}	6.08×10^{-1}

System 4.3

Consider the stiffly problem,

$y_1' = -100y_1 + 9.901y_2; \quad y_1(0) = 1$

$y_2' = 0.1y_1 - y_2; \quad y_2(0) = 10, \quad h = 0.1$

With Exact Solution

$y_1(x) = e^{-0.99x}$

$y_2(x) = 10e^{-0.99x}$

$x \in [0, 1]$

Table 3. Comparison of absolute error

X	***Absolute error***	
	$K=2, \quad p=8$	
	$y_1(x)$	$y_2(x)$
0.1	1.80×10^{-9}	1.40×10^{-8}
0.2	2.70×10^{-9}	2.30×10^{-8}
0.3	3.70×10^{-9}	3.30×10^{-8}
0.4	4.40×10^{-9}	3.90×10^{-8}
0.5	5.00×10^{-9}	470×10^{-8}
0.6	5.20×10^{-9}	5.00×10^{-8}
0.7	5.40×10^{-9}	5.20×10^{-8}
0.8	5.70×10^{-9}	5.40×10^{-8}
0.9	5.60×10^{-9}	5.50×10^{-8}
1.0	5.70×10^{-9}	5.50×10^{-8}

5. CONCLUSION

The formation of implicit backward difference Adam's Moulton formulae has been studied in this paper. We use interpolation and collocation via Cash, [1] in deriving the method. The property of the method has been analyzed, and it was found to be consistent, zero-stable and convergent with region of absolutely stability within which the method is stable. Therefore the

general solution of first order backward difference Adam's Moulton formulae is a convenient technique for determining the solutions of mathematical modeling since it can approximate the result even though the efficiency is less than the other multistep method. This study concluded that, the multistep method is very effective method for solving linear IVPs.

6. RECOMMENDATION

The pair of backward difference Adam's Moulton formulae developed in this paper is recommended for testing first order stiff system of ordinary differential equations. The basis function [1] used is also recommended for the derivation of Numerical methods for second order differential equations and the pair of methods derived are also recommended for the solution of systems of second order stiffly ordinary differential equations.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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