



Cubic Transformation of Exponential Weibull and its Statistical Properties

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Authors' contributions

This work was carried out in collaboration between both authors. Author HW designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript. Author ZB managed the analyses of the study and the literature searches. Both authors read and approved the final manuscript.

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ABSTRACT

In this paper, a cubic transformation exponential Weibull distribution is proposed by using the family of cubic transformation distributions introduced by Rahman et al. The reasoning process of the proposed cubic transformation exponential Weibull distribution is discussed in detail, and its statistical properties and parameter estimation are also discussed. Using real data, the maximum likelihood estimation is used to simulate. Through the comparison of fitting results, it is concluded that the cubic transformation exponential Weibull distribution proposed in this paper has stronger applicability.

Keywords: *Cubic transformation; maximum likelihood estimation; moment; reliability analysis; exponential Weibull.*

1. INTRODUCTION

Waloddi Weibull [1] put forward Weibull distribution in 1951, which is a common statistical model in the field of reliability analysis. Many

scholars have generalized Weibull distribution. Mudholkar [2] extended Weibull distribution to survival data analysis in 1996. Pham [3] introduced the recent development of Weibull distribution in 2007, including the improved

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Weibull distribution introduced by Xie [4] in 2002 and Lai [5] in 2003, the exponential Weibull distribution introduced by Nassar [6] in 2003 and the exponential Weibull distribution introduced by Pal [7] in 2006, and Jiang et al. [8] introduced the inverse Weibull distribution in 2001, and briefly introduced the general properties of this kind of distribution, and discussed some future research directions of this subject. In 2007, Shaw and Buckley [9] proposed a family of quadratic transformation distributions. Based on this, they gave examples of partial uniform, partial normal and partial exponential distributions, and proposed the peak variation. The distribution function of quadratic transformation distribution family is

$$F(x) = (1 + \lambda)G(x) - \lambda G^2(x) \quad (1)$$

where $\lambda \in [-1, 1]$ is the conversion parameter.

In 2011, Aryal [10] proposed the transformation Weibull distribution by using the method proposed by Shaw, which has a wider applicability in reliability analysis. As an extension, Granzotto [11] and Al Kadim [12] proposed two cubic transformation distributions in 2017 to capture the complexity of data. Rahman [13] extended the family of transformation distributions to the family of cubic transformation distributions. In 2019, Mohabubur [14] introduced the properties and applications of cubic transformation Weibull distribution, and carried out simulation research, using two real data sets to study the applicability of the proposed cubic transformation Weibull distribution.

The family of cubic transformation distributions introduced by Rahman [13] is flexible enough to capture the complexity of real life data sets. The distribution function of cubic transformation distribution family is given by the following formula

$$F(x) = (1 + \lambda_1)G(x) + (\lambda_2 - \lambda_1)G^2(x) - \lambda_2 G^3(x), x \in R \quad (2)$$

Substituting equation (3) into equation (2), the distribution function of CTEW distribution is obtained as follows:

$$F(x) = (1 + \lambda_1) \left(1 - e^{-(x/\lambda)^k}\right)^\beta + (\lambda_2 - \lambda_1) \left(1 - e^{-(x/\lambda)^k}\right)^{2\beta} - \lambda_2 \left(1 - e^{-(x/\lambda)^k}\right)^{3\beta}, x \in [0, \infty) \quad (5)$$

where $\lambda, \beta, k \in R^+$ is the scale parameter and shape parameter, $\lambda_1 \in [-1, 1], \lambda_2 \in [-1, 1]$ is the conversion parameter.

where $\lambda_1 \in [-1, 1], \lambda_2 \in [-1, 1]$ is the conversion parameter and $-2 \leq \lambda_1 + \lambda_2 \leq 1$.

The main purpose of this paper is to construct a new statistical distribution based on the study of cubic transformation Weibull distribution, using cubic transformation family formula (2), selecting exponential Weibull distribution with three parameters as benchmark distribution, adding a shape parameter on the basis of Mahabubur [14], and so on. The new model is called cubic transformation exponential Weibull distribution (CTEW). With the help of the real motor vehicle insurance personal injury loss data set, the maximum likelihood estimation is used to simulate, in order to prove that the new model constructed in this paper has better fitting effect. Empirical analysis shows that the model studied in this paper has better fitting effect.

2. STRUCTURE OF CTEW DISTRIBUTION

The distribution function of exponential Weibull is given by

$$G(x) = \left(1 - e^{-(x/\lambda)^k}\right)^\beta, x \in [0, \infty) \quad (3)$$

where $\lambda, k, \beta \in R^+$ is shape parameter and scale parameter.

According to Aryal and Tsokos [10], the method of transforming Weibull distribution is proposed. The distribution function of quadratic transformation exponential Weibull distribution is obtained by substituting formula (3) into formula (1).

$$F(x) = \left(1 - e^{-(x/\lambda)^k}\right)^\beta \left[1 + \theta - \theta \left(1 - e^{-(x/\lambda)^k}\right)^\beta\right], x \in [0, \infty) \quad (4)$$

where $\lambda, \beta, k \in R^+$ is the scale parameter and shape parameter, θ is the conversion parameter.

By deriving equation (5), the density function of CTEW distribution is obtained as follows

$$f(x) = \frac{k\beta}{\lambda^k} x^{k-1} e^{-(x/\lambda)^k} \left[(1 + \lambda_1) \left(1 - e^{-(x/\lambda)^k}\right)^{\beta-1} + 2(\lambda_2 - \lambda_1) \left(1 - e^{-(x/\lambda)^k}\right)^{2\beta-1} - 3\lambda_2 \left(1 - e^{-(x/\lambda)^k}\right)^{3\beta-1} \right] \quad (6)$$

where $\lambda, \beta, k \in R^+, \lambda_1 \in [-1, 1], \lambda_2 \in [-1, 1],$ and $-2 \leq \lambda_1 + \lambda_2 \leq 1.$

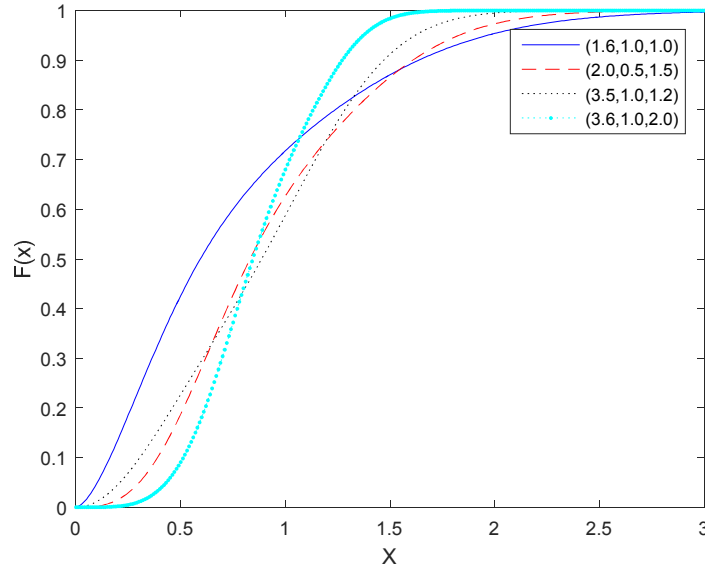


Fig. 1. Distribution function of different parameters

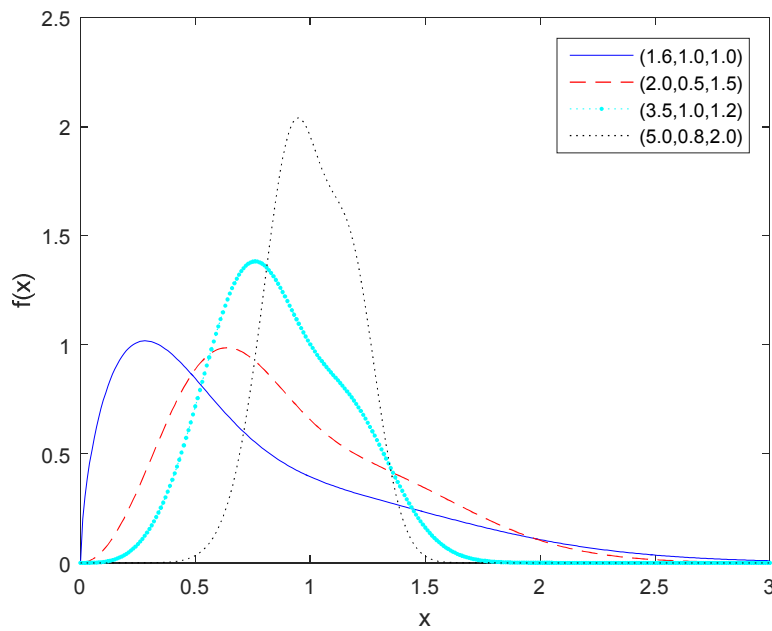


Fig. 2. Density function of different parameters

3. STATISTICAL PROPERTIES

The following section discusses the statistical properties of CTEW distribution.

3.1 Moment

Moments play an important role in the shape of distributed images. Next, we discuss the moment of CTEW distribution.

Theorem 1 The order moment of CTEW distribution is

$$E(x^r) = \beta \lambda^r \Gamma\left(\frac{r+k}{k}\right) \left[\sum_{i=0}^{\infty} \binom{\beta-1}{i} (-1)^i (1+\lambda_1) \left(\frac{1}{1+i}\right)^{\frac{r+k}{k}} + \sum_{m=0}^{\infty} \binom{2\beta-1}{m} (-1)^m 2(\lambda_2 - \lambda_1) \left(\frac{1}{1+m}\right)^{\frac{r+k}{k}} - \sum_{n=0}^{\infty} \binom{3\beta-1}{n} (-1)^n 3\lambda_2 \left(\frac{1}{1+n}\right)^{\frac{r+k}{k}} \right] \tag{7}$$

The mean and variance are given below :

$$E(x) = \beta r \Gamma\left(\frac{1+k}{k}\right) \left[\sum_{i=0}^{\infty} \binom{\beta-1}{i} (-1)^i (1+\lambda_1) \left(\frac{1}{1+i}\right)^{\frac{1+k}{k}} + \sum_{m=0}^{\infty} \binom{2\beta-1}{m} (-1)^m 2(\lambda_2 - \lambda_1) \left(\frac{1}{1+m}\right)^{\frac{1+k}{k}} - \sum_{n=0}^{\infty} \binom{3\beta-1}{n} (-1)^n 3\lambda_2 \left(\frac{1}{1+n}\right)^{\frac{1+k}{k}} \right]$$

$$V(x) = \beta \lambda^2 \left\{ \Gamma\left(\frac{2+k}{k}\right) \left[\sum_{i=0}^{\infty} \binom{\beta-1}{i} (-1)^i (1+\lambda_1) \left(\frac{1}{1+i}\right)^{\frac{2+k}{k}} + \sum_{m=0}^{\infty} \binom{2\beta-1}{m} (-1)^m 2(\lambda_2 - \lambda_1) \left(\frac{1}{1+m}\right)^{\frac{2+k}{k}} - \sum_{n=0}^{\infty} \binom{3\beta-1}{n} (-1)^n 3\lambda_2 \left(\frac{1}{1+n}\right)^{\frac{2+k}{k}} \right] - \beta \Gamma^2\left(\frac{1+k}{k}\right) \left[\sum_{i=0}^{\infty} \binom{\beta-1}{i} (-1)^i (1+\lambda_1) \left(\frac{1}{1+i}\right)^{\frac{1+k}{k}} + \sum_{m=0}^{\infty} \binom{2\beta-1}{m} (-1)^m 2(\lambda_2 - \lambda_1) \left(\frac{1}{1+m}\right)^{\frac{1+k}{k}} - \sum_{n=0}^{\infty} \binom{3\beta-1}{n} (-1)^n 3\lambda_2 \left(\frac{1}{1+n}\right)^{\frac{1+k}{k}} \right]^2 \right\}$$

Proof. The moment of order r is given by

$$E(x^r) = \int_0^{\infty} x^r f(x) dx \tag{8}$$

where $f(x)$ is given in equation (6), and substituting equation (6) into equation (8), the r order moment of CTEW distribution is obtained as follows:

$$E(x^r) = \frac{k\beta}{\lambda^k} (1+\lambda_1) \int_0^{\infty} x^{r+k-1} e^{-(x/\lambda)^k} \left(1 - e^{-(x/\lambda)^k}\right)^{\beta-1} dx + \frac{k\beta}{\lambda^k} 2(\lambda_2 - \lambda_1) \int_0^{\infty} x^{r+k-1} e^{-(x/\lambda)^k} \left(1 - e^{-(x/\lambda)^k}\right)^{2\beta-1} dx - \frac{k\beta}{\lambda^k} 3\lambda_2 \int_0^{\infty} x^{r+k-1} e^{-(x/\lambda)^k} \left(1 - e^{-(x/\lambda)^k}\right)^{3\beta-1} dx$$

By using the generalized binomial theorem, $E(x^r)$ is reduced to

$$E(x^r) = \sum_{i=0}^{\infty} \binom{\beta-1}{i} (-1)^i \frac{k\beta}{\lambda^k} (1+\lambda_1) \int_0^{\infty} x^{r+k-1} e^{-(1+i)(x/\lambda)^k} dx + \sum_{m=0}^{\infty} \binom{2\beta-1}{m} (-1)^m \frac{k\beta}{\lambda^k} 2(\lambda_2 - \lambda_1) \int_0^{\infty} x^{r+k-1} e^{-(1+m)(x/\lambda)^k} dx - \sum_{n=0}^{\infty} \binom{3\beta-1}{n} (-1)^n \frac{k\beta}{\lambda^k} 3\lambda_2 \int_0^{\infty} x^{r+k-1} e^{-(1+n)(x/\lambda)^k} dx$$

We can get it by changing of variable

$$E(x^r) = \beta\lambda^r \Gamma\left(\frac{r+k}{k}\right) \left[\sum_{i=0}^{\infty} \binom{\beta-1}{i} (-1)^i (1+\lambda_1) \left(\frac{1}{1+i}\right)^{\frac{r+k}{k}} + \sum_{m=0}^{\infty} \binom{2\beta-1}{m} (-1)^m 2(\lambda_2 - \lambda_1) \left(\frac{1}{1+m}\right)^{\frac{r+k}{k}} - \sum_{n=0}^{\infty} \binom{3\beta-1}{n} (-1)^n 3\lambda_2 \left(\frac{1}{1+n}\right)^{\frac{r+k}{k}} \right] \tag{9}$$

Let $r = 1$ in formula (9) get the mean value, and the variance can be obtained by the following formula:

$$V(X) = E(X^2) - \{E(X)\}^2$$

where $E(X^r)$, $r = 1, 2$ is obtained from equation (9).

Let $r > 2$ in (9) give the higher moment.

3.2 Moment Generating Function

Moment generating function is a useful function to obtain moments of random variables. Moment generating function of CTEW distribution is given by the following theorem.

Theorem 2 Let X obey the CTEW distribution, then the moment generating function $M_X(t)$ is given by the following formula:

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \beta\lambda^r \Gamma\left(\frac{r+k}{k}\right) \left[\sum_{i=0}^{\infty} \binom{\beta-1}{i} (-1)^i (1+\lambda_1) \left(\frac{1}{1+i}\right)^{\frac{r+k}{k}} + \sum_{m=0}^{\infty} \binom{2\beta-1}{m} (-1)^m 2(\lambda_2 - \lambda_1) \left(\frac{1}{1+m}\right)^{\frac{r+k}{k}} - \sum_{n=0}^{\infty} \binom{3\beta-1}{n} (-1)^n 3\lambda_2 \left(\frac{1}{1+n}\right)^{\frac{r+k}{k}} \right] \tag{10}$$

where $t \in R$.

Proof The moment generating function is obtained by the following formula:

$$M_X(t) = E[e^{tx}] = \int_0^{\infty} e^{tx} f(x) dx$$

where $f(x)$ is given in equation (6).

According to the expression of series e^{tx} given by Gradshteyn and Ryzhik [15], it is obtained that

$$M_X(t) = \int_0^\infty \sum_{r=0}^\infty \frac{t^r}{r!} x^r f(x) dx = \sum_{r=0}^\infty \frac{t^r}{r!} E(X^r) \tag{11}$$

Formula (10) can be obtained by substituting formula (7) $E(X^r)$ into formula (11)

3.3 Characteristic Function

The characteristic function plays a central role and completely defines the density function. The following theorem gives the characteristic function of CTEW distribution.

Theorem 3 Let X obey the CTEW distribution, and the characteristic function $\Phi_X(t)$ of X is:

$$\begin{aligned} \Phi_X(t) = & \sum_{r=0}^\infty \frac{(it)^r}{r!} \beta \lambda^r \Gamma\left(\frac{r+k}{k}\right) \left[\sum_{i=0}^\infty \binom{\beta-1}{i} (-1)^i (1+\lambda_1) \left(\frac{1}{1+i}\right)^{\frac{r+k}{k}} + \sum_{m=0}^\infty \binom{2\beta-1}{m} (-1)^m 2(\lambda_2-\lambda_1) \left(\frac{1}{1+m}\right)^{\frac{r+k}{k}} \right. \\ & \left. - \sum_{n=0}^\infty \binom{3\beta-1}{n} (-1)^n 3\lambda_2 \left(\frac{1}{1+n}\right)^{\frac{r+k}{k}} \right] \end{aligned}$$

where $i = \sqrt{-1}$ is an imaginary unit, $t \in R$.

3.4 Quantile Function

The quantile function x_q of random variable is the inverse of its distribution function. By solving equation (5), the quantile function of CTEW distribution is obtained.

$$x_q = \lambda \left[-\ln \left(1 - y^{\frac{1}{\beta}} \right) \right]^{\frac{1}{k}} \tag{12}$$

Where

$$y = -\frac{b}{3a} + \frac{1}{3a} 2^{-\frac{1}{3}} \left(\xi_2 + \sqrt{\xi_2^2 + 4\xi_1^3} \right)^{\frac{1}{3}} + \frac{1}{3a} 2^{-\frac{1}{3}} \left(\xi_2 - \sqrt{\xi_2^2 + 4\xi_1^3} \right)^{\frac{1}{3}} \tag{13}$$

$$\xi_1 = -b^2 + 3ac, \xi_2 = -2b^3 + 9abc, a = -\lambda_2, b = \lambda_2 - \lambda_1, c = 1 + \lambda_1.$$

3.5 Simulated Random Sample

Quantile function can be used to generate random data from cubic transform exponential Weibull distribution. Random data from the cubic transform exponential Weibull distribution can be obtained by using the following expression.

$$(1 + \lambda_1) \left(1 - e^{-(x/\lambda)^k} \right)^\beta + (\lambda_2 - \lambda_1) \left(1 - e^{-(x/\lambda)^k} \right)^{2\beta} - \lambda_2 \left(1 - e^{-(x/\lambda)^k} \right)^{3\beta} = u$$

Further simplification is needed

$$X = \lambda \left[-\ln \left(1 - y^{\frac{1}{\beta}} \right) \right]^{\frac{1}{k}} \tag{14}$$

where y is given in equation (13). Using (14) to give the values of model parameter $\lambda, k, \beta, \lambda_1, \lambda_2$, we can get the random samples of CTEW distribution.

3.6 Reliability Analysis

The reliability function of CTEW distribution is only a supplement to the distribution function:

$$R(t) = 1 - F(t) = 1 - (1 + \lambda_1) \left(1 - e^{-(x/\lambda)^k} \right)^\beta - (\lambda_2 - \lambda_1) \left(1 - e^{-(x/\lambda)^k} \right)^{2\beta} + \lambda_2 \left(1 - e^{-(x/\lambda)^k} \right)^{3\beta}$$

The hazard function is the ratio of density function to reliability function:

$$h(t) = \frac{\frac{k\beta}{\lambda^k} x^{k-1} e^{-(x/\lambda)^k} \left[(1 + \lambda_1) \left(1 - e^{-(x/\lambda)^k} \right)^{\beta-1} + 2(\lambda_2 - \lambda_1) \left(1 - e^{-(x/\lambda)^k} \right)^{2\beta-1} - 3\lambda_2 \left(1 - e^{-(x/\lambda)^k} \right)^{3\beta-1} \right]}{1 - (1 + \lambda_1) \left(1 - e^{-(x/\lambda)^k} \right)^\beta - (\lambda_2 - \lambda_1) \left(1 - e^{-(x/\lambda)^k} \right)^{2\beta} + \lambda_2 \left(1 - e^{-(x/\lambda)^k} \right)^{3\beta}}$$

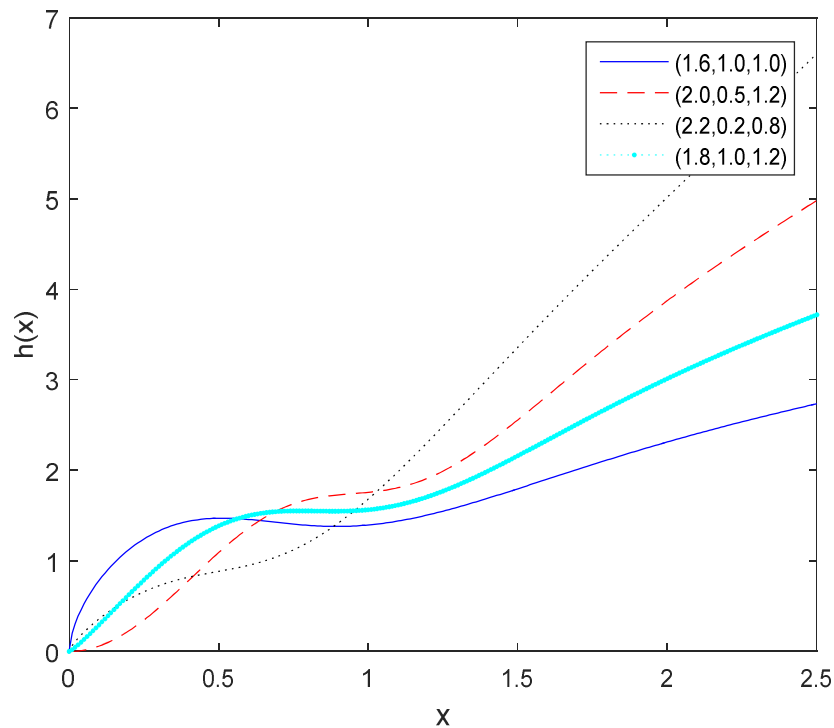


Fig. 3. Hazard function change chart of different parameters

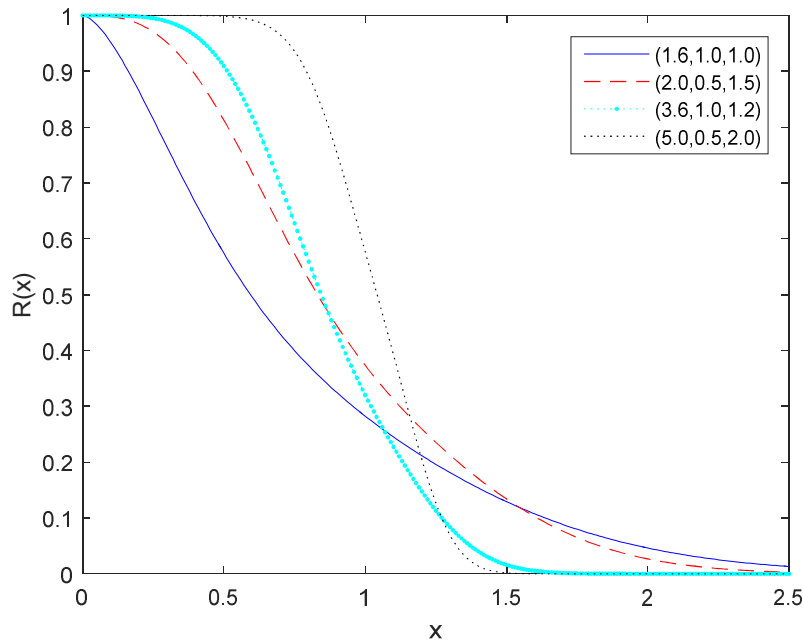


Fig. 4. Reliability function of different parameters

4. ORDER STATISTICS

The density function of r order statistics of CTEW distribution is given as follows:

$$\begin{aligned}
 f_{X_{r:n}}(x) &= \frac{n}{(r-1)!(n-r)!} [F(x)]^{r-1} [1-F(x)]^{n-r} f(x) \\
 &= \frac{n}{(r-1)!(n-r)!} \frac{k\beta}{\lambda^k} x^{k-1} e^{-(x/\lambda)^k} \left[(1 + \lambda_1 (1 - e^{-(x/\lambda)^k})^{\beta-1} + 2(\lambda_2 - \lambda_1 (1 - e^{-(x/\lambda)^k})^{2\beta-1} \right. \\
 &\quad \left. - 3\lambda_2 (1 - e^{-(x/\lambda)^k})^{3\beta-1}) \times \left[(1 + \lambda_1 (1 - e^{-(x/\lambda)^k})^\beta + (\lambda_2 - \lambda_1 (1 - e^{-(x/\lambda)^k})^{2\beta} - \lambda_2 (1 - e^{-(x/\lambda)^k})^{3\beta})^{r-1} \right] \right. \\
 &\quad \left. \times \left[1 - (1 + \lambda_1 (1 - e^{-(x/\lambda)^k})^\beta - (\lambda_2 - \lambda_1 (1 - e^{-(x/\lambda)^k})^{2\beta} + \lambda_2 (1 - e^{-(x/\lambda)^k})^{3\beta})^{n-r} \right] \right]
 \end{aligned} \tag{15}$$

where $r = 1, 2, \dots, n$, let $r = 1$ in equation (15), we obtain the density function of the minimum order statistic $X_{1:n}$ as follows:

$$\begin{aligned}
 f_{X_{1:n}}(x) &= \frac{nk\beta}{\lambda^k} x^{k-1} e^{-(x/\lambda)^k} \left[(1 + \lambda_1 (1 - e^{-(x/\lambda)^k})^{\beta-1} + 2(\lambda_2 - \lambda_1 (1 - e^{-(x/\lambda)^k})^{2\beta-1} - 3\lambda_2 (1 - e^{-(x/\lambda)^k})^{3\beta-1}) \right. \\
 &\quad \left. \times \left[1 - (1 + \lambda_1 (1 - e^{-(x/\lambda)^k})^\beta - (\lambda_2 - \lambda_1 (1 - e^{-(x/\lambda)^k})^{2\beta} + \lambda_2 (1 - e^{-(x/\lambda)^k})^{3\beta})^{n-1} \right] \right]
 \end{aligned}$$

Let $r = n$ in equation (15) obtain the density function of the largest order statistic $X_{n:n}$ as follows:

$$\begin{aligned}
 f_{X_{n:n}}(x) &= \frac{nk\beta}{\lambda^k} x^{k-1} e^{-(x/\lambda)^k} \left[(1 + \lambda_1 (1 - e^{-(x/\lambda)^k})^{\beta-1} + 2(\lambda_2 - \lambda_1 (1 - e^{-(x/\lambda)^k})^{2\beta-1} - 3\lambda_2 (1 - e^{-(x/\lambda)^k})^{3\beta-1}) \right. \\
 &\quad \left. \times \left[(1 + \lambda_1 (1 - e^{-(x/\lambda)^k})^\beta + (\lambda_2 - \lambda_1 (1 - e^{-(x/\lambda)^k})^{2\beta} - \lambda_2 (1 - e^{-(x/\lambda)^k})^{3\beta})^{n-1} \right] \right]
 \end{aligned}$$

When $\lambda_1 = \lambda_2 = 0$, the order statistics of order r are obtained as follows:

$$g_{X_{r:n}}(x) = \frac{n!k\beta}{(r-1)!(n-r)!\lambda^k} x^{k-1} e^{-(x/\lambda)^k} \left(1 - e^{-(x/\lambda)^k}\right)^{\beta r-1} \left[1 - \left(1 - e^{-(x/\lambda)^k}\right)^\beta\right]^{n-r}; r = 1, 2, \dots, k.$$

The k -moment of order statistics of CTEW distribution is given by the following formula:

$$E(X_{r:n}^k) = \int_0^\infty x_r^k \cdot f_{X_{r:n}}(x) \cdot dx$$

where $f_{X_{r:n}}(x)$ is given in equation (15).

5. PARAMETER ESTIMATION AND REASONING

In this part, we use the maximum likelihood method to estimate the parameters of the cubic transformation exponential Weibull distribution. Consider a random sample x_1, x_2, \dots, x_n obeying CTEW distribution. The likelihood function is given by

$$L = \frac{k^n \beta^n}{\lambda^{nk}} \prod_{i=1}^n x_i^{k-1} e^{-\sum_{i=1}^n (x_i/\lambda)^k} \prod_{i=1}^n \left[(1 + \lambda_1) \left(1 - e^{-(x_i/\lambda)^k}\right)^{\beta-1} + 2(\lambda_2 - \lambda_1) \left(1 - e^{-(x_i/\lambda)^k}\right)^{2\beta-1} - 3\lambda_2 \left(1 - e^{-(x_i/\lambda)^k}\right)^{3\beta-1} \right]$$

The log likelihood function $l = \ln(L)$ is

$$l = n \ln k\beta - nk \ln \lambda + (k-1) \sum_{i=1}^n \ln x_i - \sum_{i=1}^n (x_i/\lambda)^k + \sum_{i=1}^n \ln \left[(1 + \lambda_1) \left(1 - e^{-(x_i/\lambda)^k}\right)^{\beta-1} + 2(\lambda_2 - \lambda_1) \left(1 - e^{-(x_i/\lambda)^k}\right)^{2\beta-1} - 3\lambda_2 \left(1 - e^{-(x_i/\lambda)^k}\right)^{3\beta-1} \right] \tag{16}$$

The maximum likelihood estimation of $\lambda, k, \lambda_1, \lambda_2$ is obtained by maximizing the log likelihood function (16). The derivative of the unknown parameter is as follows:

If we take the derivative of l to λ and let $\beta_1 = 1 + \lambda_1, \beta_2 = \lambda_2 - 1$, we can get

$$\begin{aligned} \frac{\partial l}{\partial \lambda} &= -\frac{nk}{\lambda} + \sum_{i=1}^n \frac{kx_i}{\lambda^2} (x_i/\lambda)^{k-1} + \sum_{i=1}^n \frac{-kx_i^k e^{-(x_i/\lambda)^k} \left[\beta_1 (\beta - 1) \left(1 - e^{-(x_i/\lambda)^k}\right)^{\beta-2} + \right.}{\lambda^{k+1} \left[\beta_1 \left(1 - e^{-(x_i/\lambda)^k}\right)^{\beta-1} \right.} \\ &\quad \left. \left. 2\beta_2 (2\beta - 1) \left(1 - e^{-(x_i/\lambda)^k}\right)^{\beta-2} - 3\lambda_2 (3\beta - 1) \left(1 - e^{-(x_i/\lambda)^k}\right)^{3\beta-2} \right] \right]}{+ 2\beta_2 \left(1 - e^{-(x_i/\lambda)^k}\right)^{2\beta-1} - 3\lambda_2 \left(1 - e^{-(x_i/\lambda)^k}\right)^{3\beta-1}} \\ \frac{\partial l}{\partial k} &= \frac{n}{k} - n \ln \lambda + \sum_{i=1}^n \ln x_i - \sum_{i=1}^n (x_i/\lambda)^k \ln(x_i/\lambda) + \\ &\quad \sum_{i=1}^n \frac{(x_i/\lambda)^k \ln(x_i/\lambda) e^{-(x_i/\lambda)^k} \left[\beta_1 (\beta - 1) \left(1 - e^{-(x_i/\lambda)^k}\right)^{\beta-2} + 2\beta_2 (2\beta - 1) \left(1 - e^{-(x_i/\lambda)^k}\right)^{2\beta-2} - 3\lambda_2 (3\beta - 1) \left(1 - e^{-(x_i/\lambda)^k}\right)^{3\beta-2} \right]}{\beta_1 \left[\left(1 - e^{-(x_i/\lambda)^k}\right)^{\beta-1} \right] + 2\beta_2 \left(1 - e^{-(x_i/\lambda)^k}\right)^{2\beta-1} - 3\lambda_2 \left(1 - e^{-(x_i/\lambda)^k}\right)^{3\beta-1}} \\ \frac{\partial l}{\partial \lambda_2} &= \frac{2 \left(1 - e^{-(x_i/\lambda)^k}\right)^{2\beta-1} - 3 \left(1 - e^{-(x_i/\lambda)^k}\right)^{3\beta-1}}{\beta_1 \left(1 - e^{-(x_i/\lambda)^k}\right)^{\beta-1} + 2\beta_2 \left(1 - e^{-(x_i/\lambda)^k}\right)^{2\beta-1} - 3\lambda_2 \left(1 - e^{-(x_i/\lambda)^k}\right)^{3\beta-1}} \end{aligned}$$

$$\frac{\partial}{\partial \lambda_1} = \frac{\left(1 - e^{-(x_i/\lambda)^k}\right)^{\beta-1} - 2\left(1 - e^{-(x_i/\lambda)^k}\right)^{2\beta-1}}{\beta_1\left(1 - e^{-(x_i/\lambda)^k}\right)^{\beta-1} + 2\beta_2\left(1 - e^{-(x_i/\lambda)^k}\right)^{2\beta-1} - 3\lambda_2\left(1 - e^{-(x_i/\lambda)^k}\right)^{3\beta-1}}$$

6. EMPIRICAL ANALYSIS

This paper uses a group of real data of personal injury claims in motor vehicle insurance for empirical analysis, see literature [16]. Individual extreme values are excluded from this set of data, and the remaining claims amount to 2746 in US dollars. The relevant descriptive statistics of the data are shown in Table 1. The maximum loss is about 120000, and there are only two claims with a skewness of about 6, indicating that the data has a long right tail, and the kurtosis reaches 54, indicating that the distribution of the data is steep.

Cubic transform exponential Weibull distribution (CTEW) and exponential distribution (E), Weibull distribution, gamma distribution, permutation Weibull distribution (TW), exponential Weibull distribution (EW), Weibull exponential distribution (WE), Weibull inverse exponential distribution (WIE) and permutation exponential distribution (TGE) were used to fit the data. The model parameter estimation results are shown in Table 2.

The goodness of fit indexes include NLL, AIC, BIC, HQIC and CAIC. Where NLL is the maximum likelihood estimation of the log likelihood function, and other indicators are defined as follows:

$$AIC = 2NLL + 2q$$

$$BIC = 2NLL + q \log(n)$$

$$CAIC = 2NLL + \frac{2qn}{n - q - 1}$$

$$HQIC = 2NLL + 2q \log(\log(n))$$

where q represents the number of parameters and n represents the sample size. The results of fitting different models by MATLAB and R language are given in Table 3.

The fitting results show that the CTEW model proposed in this paper fits best with NLL, AIC, BIC, HQIC and CAIC as indexes.

Table 1. Descriptive statistics of personal injury loss data in motor vehicle insurance

Statistics	The value of Statistics
Mean	5669.724566
Median	4107.5
Maximum	123247.1
Minimum	30.27728
Skewness	6.140982
Kurtosis	54.10929
Standard deviation	7759.067
1 / 4 Quantile	2641
3 / 4 Quantile	5727.75

Table 2. Estimation results of model parameters

Distribution	Estimate					
	λ	k	α	β	λ_1	λ_2
E	0.0002	-	-	-	-	-
Weibull	5909.4031	1.0906	-	-	-	-
Gamma	3761.1040	1.5075	-	-	-	-
TW	-1.0000	-	1.0667	-0.1925	-	-
EW	320.4961	0.4529	15.2886	-	-	-
WE	0.0000002	-	1240.6860	1.0866	-	-
WIE	2501.5480	-	0.6375	0.8636	-	-
TGE	0.9000	-	-1.0000	-0.0006	-	-
CTEW	35.6978	0.3373	-	34.9150	-0.7646	-1.0000

Table 3. Model fitting index results

Model	Goodness of fit index				
	NLL	AIC	CAIC	HQIC	BIC
E	26479.4	52960.8	52960.781	52962.918	52966.698
Weibull	26456.5	52917.0	52916.984	52921.257	52928.816
Gamma	26354.5	52713.0	52713.004	52711.069	52724.836
TW	30849.1	61704.2	61704.209	61710.615	61721.954
EW	26008.7	52023.4	52023.389	52029.795	52041.134
WE	26462.1	52930.2	52930.189	52936.595	52947.934
WIE	26313.8	52633.5	52633.509	52639.915	52651.254
TGE	26312.3	52630.5	52630.529	52636.935	52648.274
CTEW	25940.1	51890.1	51890.122	51900.791	51919.689

7. CONCLUSION

In this paper, a new kind of CTEW distribution is proposed by using the general method of cubic transformation distribution family. By using the generalized Newton binomial expansion, the expressions of distribution function, density function, hazard function, quantile, moment, moment generating function, characteristic function and ordinal statistics of the new model are given. By using R language to simulate and analyze the personal injury loss data of a kind of motor vehicle insurance, the model parameter estimation results and goodness of fit comparison are given. The results show that the model proposed in this paper has the best fitting results under various criteria. The benchmark distribution can be taken as permutation distribution, gamma distribution, Weibull index distribution and so on, and the corresponding structural properties can be obtained by using similar research methods. This kind of new model will have great advantages in some specific data fitting.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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