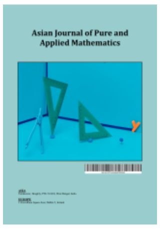

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Properties of Isosceles, Pythagorean and Isosceles-Pythagorean Vectors in the Characterization of Hilbert Spaces

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

All Hilbert spaces are Banach spaces but the converse is not necessarily true. Characterization of Banach spaces as Hilbert spaces has had different approaches for various Banach spaces. It has been shown that a separable Banach space which is almost transitive with vector orthogonalities for dimension greater than three is a Hilbert space. It worthy to note that micro transitivity together with Isosceles (I), Pythagorean (P) and Isosceles Pythagorean (IP) orthogonalities in the unit sphere have some essential properties that can be considered in characterization of Hilbert spaces. In this study, separable micro transitive Banach spaces are examined and their characterization as Hilbert spaces is achieved by applying the I-vector property in affine sets along with the P and IP-vector properties. In particular, by letting a separable Banach space X of $dimX \geq 2$ possessing micro transitivity property with I, P, and IP vectors, then X is a Hilbert space. The results of this research are expected to be useful in algebra and differential operators, particularly for calculating wave functions and formulation of theory.

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1 Introduction

All Hilbert spaces are Banach spaces but the converse is not necessarily true. In order to find the converse relation, different approaches have been employed. [1] characterized general Banach Spaces as Hilbert spaces by the use of Parallelogram law. However, this is not satisfied by all Banach spaces. This problem is closely related to Banach Mazur rotation problem which ask whether if each transitive and separable Banach spaces are Hilbert spaces. There are various types of transitivity in literature such as Almost transitive, convex transitive, asymptotic transitive and micro transitive.

Randrianantoanina [2] characterized almost transitive Banach space as Hilbert space if it contains a subspace of codimension one which is 1-complemented. [3] characterized Banach space as Hilbert space if it is a non-empty homogeneous direction set of isosceles orthogonality in a unit sphere. However, the Banach space was not classified as whether transitive, almost transitive or micro transitive. [4] used the concept of asymptotic transitivity which also yields some new properties about classical almost transitive spaces and characterized the space as Hilbert space. However, the characteristics of the vectors in the space were not given.

Guerrero et al. [5] characterized real Banach spaces that are almost transitive and have isometric reflection as Hilbert spaces. [6] characterized almost transitive Banach spaces as Hilbert spaces if they contain I- and IP-vectors. [7] proved that one dimensional Lebesgue (L_p) space which is uniformly micro transitive is a Hilbert space if $p = 2$. The characteristics of the vectors in the space were not given and for one or more dimensional Lebesgue spaces and $p > 2$ were not characterized as Hilbert spaces. [8] used properties of homogeneous and additivity of isosceles and Pythagorean orthogonality respectively to characterize normed linear spaces as inner product spaces, however these properties are not satisfied by all Banach spaces and therefore the (L_p) space was not classified as transitive, almost transitive or micro transitive.

Dragomir et al. [9] proved that if a Banach space which has Hermite Hadamard (HH)-P orthogonality is homogeneous then is a Hilbert space, however other properties and transitivity of the space were not given. Moreover, for a separable micro transitive Banach space which contains I, P and IP-vectors, it has not been characterized as a Hilbert space. This research therefore determines characterization of a separable micro transitive Banach space which contains an IP-vector as a Hilbert space.

The following are essential definitions that shall be utilized in achieving our main results.

Definition 1.1: [10] Micro transitive. A normed linear space X is said to be micro transitive if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $x, y \in S \subset X$ (where S is the unit sphere) satisfying $\|y-x\| \leq \delta$ there is $T \in Iso(X)$ such that $y = Tx$ and $\| T - Ix \| \leq \varepsilon$.

Definition 1.2: [11] Isosceles orthogonal. Let x and y be two vectors on a normed linear space x is said to be isosceles orthogonal to y (denoted by $x \perp_I y$) if the equality $\| x + y \| = \| x - y \|$ hold.

Definition 1.3: [6] I-vector. A unit vector x of a normed linear space X is said to be an I vector if there exist homogeneous hyperplane H such that $x \perp_I S \cap H$ (i.e., x is isosceles orthogonal to every vector from $S \cap H \subset X$, where S is unit sphere).

Definition 1.4: [12] Hyperplane. A hyperplane of a normed linear space X is any proper closed linear subset M which is not properly contained in a proper linear subset of X, or any translation $x + M \; \forall \; x \in X$ of such linear subset of M.

Definition 1.5: [13] Affine set. Let x and y be two different vectors in a normed space X, a subset $M \subset X$ is said to be affine set if $\forall x, y \in M$ a line joining x and y is contained in M.

Definition 1.6: [6] IP-vector. A unit vector x is said to be an IP-vector if for each unit vector y isosceles orthogonal to x, the equality $\| x + y \| = \| x - y \| = \sqrt{2}$ hold (or equivalently, y and -y are both Pythagorean orthogonality to x).

Definition 1.7: [8]: Pythagorean orthogonal. Let x and y be two vectors on a normed linear space X, then x is said to be Pythagorean orthogonal to y if the equality $\| x - y \|^2 = \| x \|^2 + \| y \|^2$ hold (denoted by $x \perp_P y$).

Definition 1.8: [6]P-vector. Let x be a unit vector in normed liner space X, if there exist a homogeneous hyperplane H_x such that $x \perp_p S \cap H_x$, then x is a P-vector (where $S \cap H_x \subset X$ and S is unit sphere).

2 Methodology

In achieving the results, our main tool is the following lemma.

Lemma 2.1 [3]: Let X be a Banach space with $dimX \geq 2$. Then, X is a Hilbert space if and only if it is a non-empty relative interior set of homogeneous direction of isosceles orthogonality in a unit sphere.

Properties of isosceles orthogonality such as existence, homogeneous and uniqueness and characteristic of I-vector are useful which are summarized in the following lemmas.

Lemma 2.2: [14]. Let X be a normed plane, x a point in $X \setminus \{0\}$ and M_x the length of the maximal line segment contained in sphere(S) of X and parallel to the line passing through -x and x (when there is no such segment, M_x is the set 0). Then for each number $\gamma \in [0,2 \| x \| / M_x$ ($\gamma \in [0, +\infty]$, when $M_x = 0$) there exist a unique point $y \in \gamma S$ (except of sign) such that $x \perp_I y$.

Lemma 2.3: [15]. Let X be a Banach space, then there exists a fixed constant $\alpha \neq 0, \pm 1$ such that $\| x + y \| = \| x - y \|$ implies $\| x + \alpha y \| = \| x - \alpha y \|$.

Lemma 2.4: [6]. Let X be normed space if $x \in S$ is an I-vector and H is homogeneous hyperplane such that $x \perp_I S \cap H$, then each unit vector z satisfying $x \perp_I z$ belong to H (S is unit sphere).

Lemma 2.5: [6]. Let X be a normed space and $x \in S$ is an I-vector. Then, for each $T \in G$,$T(x)$ is also an I-vector (where S is the unit sphere).

Lemma 2.6: [6]. Let X be a normed space, a unit vector $x \in X$, and P the set of all unit vector which have isosceles orthogonality to x. Then the span $(\{x\} \cup P) = X$.

This study aimed to extend the following lemma to micro transitive spaces.

Lemma 2.7: [6] Let X be a Banach space with $dimX \geq 3$. If X is almost transitive and contain I-vector x, then X is Hilbert space.

Characteristic of P-vector are useful and they are presented in lemmas 2.8 to 2.11.

Lemma 2.8: [16] Pythagorean orthogonality relation is unique.
Lemma 2.9: Let X be a normed space : $x \in X$ and C the set of all unit vectors which have Pythagorean orthogonality to X. Then the span $(\{x\} \cup C) = X$

Proof

Suppose that span $(\{x\} \cup C) \neq X$ this implies that there exist $y \in X \setminus$span $(\{x\} \cup C)$ there exist $z \in$ span $(\{x, y\} \cap C)$ which has Pythagorean orthogonality with x but $\{x, y\} \cap C = y \Rightarrow z = y$ this implies that $y \in$ span $(\{x\} \cup C)$ thus contradicting our supposition. Thus span $(\{x\} \cup C) = X$.

Lemma 2.10: Let X be a normed space and x a P-vector in X. Then, for each linear isometry $T \in G$, $T(x)$ is also a P-vector.

Proof

Let C denote the set of unit vector which have Pythagorean orthogonality with x. Let z be an arbitrary unit vector which has Pythagorean orthogonality with $T(x)$ and suppose that is micro transitive space. Then for each point $z \in S \cap T(C)$, implies that $y = T^{-1}(z) \in S \cap C$

$$\| Tx - z \|^2 = \| x - T^{-1}z \|^2$$
$$= \| x \|^2 - \| x \| \| T^{-1}z \| - \| T^{-1}z \| \| x \| + \| T^{-1}z \|^2$$
$$= \| x \|^2 + \| T^{-1}z \|^2.$$

Since T is a linear isometry then

$$\| T(x - y) \|^2 = \| T(x) - T(y) \|^2$$
$$= \| T(x) \|^2 + \| T(y) \|^2$$

This implies that $T(x)$ has Pythagorean orthogonality with z ■

Lemma 2.11: Let X be a normed linear space. If $x \in S$ is a P-vector and C is a homogenous Hyperplane such that $x \perp_P S \cap C$. Then each unit vector z satisfying $x \perp_P z$ belong to C

Proof

Suppose that there exists a unit vector in $z \notin C$ satisfying $x \perp_P z$ from lemma 2.4 it implies that span $(\{x\} \cup z \cup C) = X$ and from the uniqueness of Pythagorean orthogonality it implies that $x \perp_P \alpha x + z$ thus $\alpha x + z \in S \cap C$ since it is also a P-vector this implies that $z \in C$ which contradict thus $z \in C$.

3 Results and Discussion

Theorem 3.1 Let X be separable Banach space of $dim X \geq 2$. Then X is Hilbert space if it is micro transitive and has an I- vector

Proof

Suppose that X have I-vector and it is not micro transitive by lemma 2.2 $\exists\ x, y \in S \subset X$ such that $x \perp_I y$. We need to show that the I-vector is closed. Suppose I-vectors is finite or empty then it implies that it is closed. Let $\{x_n\}_{n=1}^{\infty}$ be sequence of I-vectors converging to a vector x that is, $\lim_{n\to\infty} x_n = x$. Since $x \in S$, let P denote the set of all unit vector that are orthogonal to x from lemma 2.6, $span(\{x\} \cup\ P) = X$ thus $span\ P$contains homogeneous hyperplane. Let H_n be a homogenous hyperplane satisfying $x_n\ \perp_I S \cap\ H_n$.this implies that $S \cap H_n\ \in P$, there exist $y_n\ \in H_n$. let $\alpha_n, \beta_n\ \in \mathbb{R}$ be two sequence which converge that is $\lim_{n\to\infty} \alpha_n = \alpha$, $\lim_{n\to\infty} \beta_n = \beta$

This implies that $x_n \perp y_n$ and since P contain homogeneous hyperplane which have vectors orthogonal to x this implies that $\exists\ \| \alpha_n x_n + \beta_n y_n \|$

Let $u = \alpha_n x_n + \beta_n y_n$

|This implies that

$$\| u \| = \| \alpha_n x_n + \beta_n y_n \|.$$

Taking the limit on both side we have

$$\lim_{n\to\infty} \| u \| = \lim_{n\to\infty} \| \alpha_n x_n + \beta_n y_n \|$$
$$\Longrightarrow \| u \| = \| \lim_{n\to\infty} \alpha_n x_n + \lim_{n\to\infty} \beta_n y_n \|$$
$$\Longrightarrow \| u \| = \| \alpha x + \beta y \|.$$

Let $\beta = 1$ this implies that $\| u \| = \| \alpha x + y \|$ and since isosceles orthogonality has the existence and uniqueness properties it implies that $x \perp_I \alpha x + y$ thus the sequence y_n converges in P thus I-vector is closed. If the I-vector is closed then it implies that there exist affine set in the sphere which is contained in P or in the $span\ P$. Let D denote the affine set. Since isosceles orthogonality is homogeneous by lemma 2.3 and it is first category it implies that the relative interior of D is empty which contradict the existence of isosceles orthogonality thus x has to be micro transitive and since relative interior is not empty by lemma 2.1 it follows that X is a Hilbert space■

Theorem 3.2: Let X be a separable Banach space of $dimX \geq 2$. If X is micro transitive and contains a P-vector then it is a Hilbert space.

Proof

Suppose that x is a P-vector in X this implies that there exist $y \in X$ such that both y and $-y$ have Pythagorean orthogonality with x

$$\| x + y \|^2 = \| x \|^2 + \| y \|^2$$
$$\Longrightarrow \| x \|^2 + \| y \|^2.$$

We need to show that P-vector is closed. Suppose that the set of p-vectors is finite or empty then it is closed. Let $(x_n)_{n=1}^{\infty}$ be an arbitrary converging sequence contained in the P-vector such that $\lim_{n\to\infty} x_n = x$. Let y be an arbitrary unit vector Pythagorean orthogonal to x. Let C denote the set of unit vector which have Pythagorean orthogonal to x this implies that $\exists\ C_n \subset C$ such that $y_n \in C_n$ suppose that C_n is a homogenous hyperplane which satisfies $x_n \perp_P S \cap C_n$ this implies that $S \cap C_n \subset C$. Let $\alpha_n, \beta_n \in \mathbb{R}$ be two sequence which converge that is, $\lim_{n\to\infty} \alpha_n = \alpha$, $\lim_{n\to\infty} \beta_n = \beta$.

This implies that $x_n \perp y_n$ and let C be an affine set this implies that

$$\exists \quad \| \alpha_n x_n + \beta_n y_n \|$$
$$\text{Let } u = \alpha_n x_n + \beta_n y_n$$
$$\Longrightarrow \| u \| = \| \alpha_n x_n + \beta_n y_n \|.$$

Taking the limit on both side we have

$$\lim_{n\to\infty} \| u \| = \lim_{n\to\infty} \| \alpha_n x_n + \beta_n y_n \|$$
$$\Longrightarrow \| u \| = \| \lim_{n\to\infty} \alpha_n x_n + \lim_{n\to\infty} \beta_n y_n \|$$
$$\Longrightarrow \| u \| = \| \alpha x + \beta y \|.$$

Let $\beta = 1$ this implies that $\| u \| = \| \alpha x + y \|$ and since Pythagorean orthogonality has uniqueness property it implies that $x \perp_P \alpha x + y$ thus the sequence y_n converges in C thus P-vector is closed

$$\| x - y \|^2 = \| x \|^2 + \| y \|^2$$
$$\Longrightarrow \| x - (\alpha x + y) \|^2 = \| x \|^2 + \| \alpha x + y \|^2.$$
$$\text{Let } F(\alpha) = \| x - (\alpha x + y) \|^2 - \| x \|^2 - \| \alpha x + y \|^2.$$

Suppose that $\alpha = 0$

$$\Longrightarrow F(0) = \| x - y) \|^2 - \| x \|^2 - \| y \|^2 \geq 0$$
$$\Longrightarrow F(1) = \| y) \|^2 - \| x \|^2 - \| x + y \|^2 \leq 0.$$

Since $\mathrm{F}(0) \geq 0$ and $\mathrm{F}(1) \leq 0$ it implies that there exist $\alpha \in [0,1]$ such that $\mathrm{F}(\alpha) = 0$.

Suppose that $\mathrm{F}(0) < 0$.

If $\mathrm{F}(-1) \geq 0$ then there exist α in the interval $[-1,0]$ such that $\mathrm{F}(\alpha) = 0$.

If $\mathrm{F}(-1) = \| 2x - y \|^2 - \| x \|^2 - \| x - y \|^2 < 0$ then

$$\mathrm{F}(-2) = \| 3x - y \|^2 - \| x \|^2 - \| 2x - y \|^2$$
$$\Rightarrow \mathrm{F}(-2) = \| 3x - y \|^2 - 2 \| x \|^2 - \| x - y \|^2$$
$$> \| 3x - y \|^2 - 3 \| x \|^2 - \| y \|^2$$
$$\geq \| 3x \|^2 - 2 \| 3x \| \cdot \| y \| + \| y \|^2 - 3 \| x \|^2 - \| y \|^2 \geq 0.$$

And so there exist α in the interval [-2, -1] such that $\mathrm{F}(\alpha) = 0$.

For $\| x + y \|^2 = \| x \|^2 + \| y \|^2$

$$\Rightarrow \| x + (\alpha x + y) \|^2 = \| x \|^2 + \| \alpha x + y \|^2$$
$$\Rightarrow \mathrm{F}(\alpha) = \| x + (\alpha x + y) \|^2 - \| x \|^2 - \| \alpha x + y \|^2.$$

If $\mathrm{F}(1) \geq 0$ then there exist α in the interval $[1,0]$ such that $\mathrm{F}(\alpha) = 0$.

If $\mathrm{F}(1) = \| 2x + y \|^2 - \| x \|^2 - \| x + y \|^2 < 0$ then

$$\mathrm{F}(-2) = \| 3x + y \|^2 - \| x \|^2 - \| 2x + y \|^2$$
$$\Rightarrow \mathrm{F}(-2) > \| 3x + y \|^2 - 2 \| x \|^2 - \| x + y \|^2$$
$$> \| 3x + y \|^2 - 3 \| x \|^2 - \| y \|^2$$
$$\geq \| 3x \|^2 + 2 \| 3x \| \cdot \| y \| + \| y \|^2 - 3 \| x \|^2 - \| y \|^2 \geq 0.$$

And so there exist an interval [2,1] such that $\mathrm{F}(\alpha) = 0$. Let α^* be any real zero of the function $\mathrm{F}(\alpha)$. Let $z = y + \alpha^* x$ so that

$$\| x - z \|^2 = \| x \|^2 + \| z \|^2$$
$$\Rightarrow \| x - y \|^2 + \| x + y \|^2 = \| x - z + \alpha^* x \|^2 + \| x + z - \alpha^* x \|^2$$
$$= \| (1 + \alpha^*)x - z \|^2 + \| (1 - \alpha^*)x + z \|^2$$
$$= \| x(1 - \alpha^*) \|^2 + \| z \|^2 + \| (1 + \alpha^*)x \|^2 + \| z \|^2$$
$$= (1 - \alpha^*)^2 \| x \|^2 + \| z \|^2 + (1 + \alpha^*)^2 \| x \|^2 + \| z \|^2$$
$$= (1 - 2\alpha^* + (\alpha^*)^2) \| x \|^2 + \| z \|^2 + (1 + 2\alpha^* + (\alpha^*)^2) \| x \|^2 + \| z \|^2$$
$$= 2 \| x \|^2 + 2\left(\alpha^{*2} \| x \|^2 + \| z \|^2\right)$$
$$= 2 \| x \|^2 + 2 \| y \|^2 \quad \blacksquare$$

Corollary 3.3

Let X be a separable Banach space of dimension greater than two. If X is micro transitive and contain an IP-vector then it is a Hilbert space.

Proof

Suppose that x is an IP-vector in X this implies that there exist $y \in X$ such that x is isosceles orthogonal to both y and $-y$ have Pythagorean orthogonality with x

$$\Rightarrow \| x + y \| = \| x - y \|$$
$$\Rightarrow \| x - y \|^2 = \| x \|^2 + \| y \|^2.$$

Since $\| x + y \| = \| x - y \|$, taking square on both side

$$\| x+y \|^2 = \| x-y \|^2$$
$$\Rightarrow (\| x+y \|)(\| x+y \|)$$
$$(\| x+y \|)(\| x+y \|) \leq (\| x \| + \| y \|)(\| x \| + \| y \|)$$
$$\leq \| x \|^2 + \| x \| \| y \| + \| y \| \| x \| + \| y \|^2.$$

Since x has isosceles orthogonality with y then their dot product is zero implying that

$$\| x-y \|^2 = \| x+y \|^2 \leq \| x \|^2 + \| y \|^2$$

Considering the equality side, it implies that Pythagorean orthogonality is well defined in the space thus x is a P-vector. Since x is an IP-vector it implies that it has isosceles orthogonality and from Lemma 2.1 and Theorem 3.1 it follows that it is a Hilbert space. ■

4 Conclusion

The main goal of the study was to characterize Banach spaces as Hilbert spaces. In this study, separable micro transitive Banach spaces were considered and by utilizing the I-vector property in affine set as well as the P and IP-vector properties, these spaces were characterized as Hilbert spaces.

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Competing Interests

Authors have declared that no competing interests exist.

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