Journal of Engineering Research and Reports

20(7): 117-137, 2021; Article no.JERR.69065
ISSN: 2582-2926

Utilization of the Karnaugh Map in Exploring Cause-effect Relations Modeled by Partially-defined Boolean Functions

Ali Muhammad Ali Rushdi[1*] and Raid Salih Badawi[1]

[1]Department of Electrical and Computer Engineering, King Abdulaziz University, P.O.Box 80200, Jeddah 21589, Saudi Arabia.

Authors' contributions

This work was carried out in collaboration between both authors. Author AMAR designed the study, performed the analysis, solved the example, contributed to the literature search, and wrote the preliminary manuscript. Author RSB managed the literature search, and helped in solving the example. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JERR/2021/v20i717348
Editor(s):
(1) Dr. Guang Yih Sheu, Chang-Jung Christian University, Taiwan.
Reviewers:
(1) S. Thangalakshmi, Indian Maritime University, India.
(2) Ooi Chek Yee, Universiti Tunku Abdul Rahman (UTAR), Malaysia.
Complete Peer review History: http://www.sdiarticle4.com/review-history/69065

Original Research Article

Received 20 March 2021
Accepted 23 May 2021
Published 24 May 2021

ABSTRACT

This paper utilizes a modern regular and modular eight-variable Karnaugh map in a systematic investigation of cause-effect relationships modeled by partially-defined Boolean functions (PDBF) (known also as incompletely specified switching functions). First, we present a Karnaugh-map test that can decide whether a certain variable must be included in a set of supporting variables of the function, and, otherwise, might enforce the exclusion of this variable from such a set. This exclusion is attained via certain don't-care assignments that ensure the equivalence of the Boolean quotient w.r.t. the variable, and that w.r.t. its complement, i.e., the exact matching of the half map representing the internal region of the variable, and the remaining half map representing the external region of the variable, in which case any of these two half maps replaces the original full map as a representation of the function. Such a variable exclusion might be continued w.r.t. other variables till a minimal set of supporting variables is reached. The paper addresses a dominantly-unspecified PDBF to obtain all its minimal sets of supporting variables without resort to integer programming techniques. For each of the minimal sets obtained, standard map methods for

**Corresponding author: Email: arushdi@yahoo.com;*

extracting prime implicants allow the construction of all irredundant disjunctive forms (IDFs). According to this scheme of first identifying a minimal set of supporting variables, we avoid the task of drawing prime-implicant loops on the initial eight-variable map, and postpone this task till the map is dramatically reduced in size. The procedure outlined herein has important ramifications for the newly-established discipline of Qualitative Comparative Analysis (QCA). These ramifications are not expected to be welcomed by the QCA community, since they clearly indicate that the too-often strong results claimed by QCA adherents need to be checked and scrutinized.

Keywords: Partially-defined boolean function; set of supporting variables; Karnaugh map; prime implicant; irredundant disjunctive form; minimal sum; complete sum.

1. INTRODUCTION

Boolean or logical analysis of data has been a hot topic of research in the past few decades [1-7]. Of a particular interest herein is a seminal paper, in which Crama et al. [1] dealt with the problem of "identifying the small subsets of plausible causes of a given effect, among a large set of factors including all the potential causes, along with many other (irrelevant) factors." They described all the combinations of possible causes which can be identified on the basis of a limited number of observations. They warned that "when only partial observations are available, no method can provide definite answers" though they hoped that even then the cause-effect relationship can be narrowed down sufficiently to provide substantial guidance. A suitable framework for handling the problem under consideration is that of partially-defined Boolean functions, also called incompletely-specified Boolean functions [1,8-22]. The computational problems arising in this framework (such as deriving minimal sets of supporting variables or finding all irredundant sums for each of these sets) are NP-hard, but in most applications of interest, the number of explanatory factors and observations are expected to be reasonably small. Crama et al. [1] mentioned similarities of their problem to similar formulations in artificial intelligence and machine learning [23-33]. Their problem is also of a nature quite similar to that addressed by Qualitative Comparative Analysis (QCA) [34-43], but it has subtle differences with the problem of digital-circuit design [41].

Crama et al. [1] addressed a problem of eight presumed variables. Out of 256 configurations in their problem, only 7 configurations were observable, and hence they dealt with a Boolean function that is partially defined. Employing integer linear programming, they found that their problem has eight minimal sets of supporting variables, and for each of these minimal sets, they obtained (usually) several irredundant disjunctive forms (IDFs). Any of these numerous IDFs (23 in total) could be the correct solution of the given problem. Though the work in [1] has been highly cited (315 citations in Google Scholar as of May 2021), we believe that it did not produce the impact it ought to. In fact, it has not been applied as the standard method for solving QCA problems except in very few recent papers [41-43]. There are possibly two main reasons for this state of affairs. One reason is that mainstream QCA typically seeks a more powerful, highly appealing and less ambiguous (albeit unjustified) result. It prefers the minimal sum among the set of all possible resulting IDFs, and hence the question of selecting a set of supporting variables is addressed by it, if ever, only as an afterthought. The second reason is that the work in [1] did not use the standard language and tools of Boolean analysis (as used by the digital-design community). In this paper, we hope to address the issue of this second reason by offering a complete exposition of the main result in [1] in standard language, aided by the pictorial insight offered by the Karnaugh map. We obtain all minimal sets of supporting variables for the problem in [1] without resort to integer programming techniques. For each of the minimal sets obtained, we employ standard map methods for extracting prime implicants to construct all irredundant disjunctive forms (IDFs). According to this scheme of first identifying a minimal set of supporting variables, we avoid the task of drawing prime-implicant loops on the initial eight-variable map, and postpone this task till the map is dramatically reduced in size.

The organization of the rest of this paper is as follows. Section 2 presents the problem of Crama et al. [1] used as a current example herein. Section 3 discusses the construction and use of the Karnaugh map. It also introduces the regular and modular version of the Karnaugh map to be used herein. This map version can be (theoretically) extended to an arbitrary large number of variables, and includes all maps of smaller sizes as special cases. Section 4 revisits

the problem in [1] from a map perspective, and recovers all the results obtained in [1]. Section 5 concludes the paper.

2. A RUNNING EXAMPLE

In Table 1, we present the running example to be used throughout this paper, which is taken from Carma, et al. [1]. Table 1 is a truth table with a single endogenous factor (outcome) f and a set of exogenous factors $(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)$ renamed herein as $(Y_3, Y_2, Y_1, Y_0, Z_3, Z_2, Z_1, Z_0)$. The renaming is made to facilitate the reconstruction of the truth table as a Karnaugh map. The set of eight factors is split into two distinct sets of Variable $Y = (Y_3, Y_2, Y_1, Y_0)$ and $Z = (Z_3, Z_2, Z_1, Z_0)$, which serve as the horizontal and vertical variables of the Karnaugh map, respectively. Each row in Table 1 represents a unique configuration that is characterized by a binary number $(Y_3Y_2Y_1Y_0Z_3Z_2Z_1Z_0)_2$ or a hexadecimal number $(H_1H_0)_{16}$, where

$$H_1 = 2^3Y_3 + 2^2Y_2 + 2^1Y_1 + 2^0Y_0 = 8Y_3 + 4Y_2 + 2Y_1 + Y_0, \quad (1)$$

$$H_0 = 2^3Z_3 + 2^2Z_2 + 2^1Z_1 + 2^0Z_0 = 8Z_3 + 4Z_2 + 2Z_1 + Z_0\,. \quad (2)$$

Note that the hexadecimal number system uses 16 symbols, the first 10 of which (0,1,2,…,9) are borrowed (with the same meaning) from the conventional decimal system, while the remaining 6 symbols are the beginning upper-case letters of the alphabet A, B, C, D, E, and F used to designate the values 10, 11, 12, 13, 14, and 15, respectively, in decimal, which correspond to 1010,1011,1100,1101,1110 and 1111, respectively in binary (Table 2). Designation of a configuration in Table 1 by a hexadecimal number H_1H_0 considerably facilitates the conversion of the truth table in Table 1 to the Karnaugh map in Fig. 1. The configuration H_1H_0 is simply located at the horizontal coordinate H_0 and the vertical coordinate H_1. Note that the map input is given in terms of the shown designations of the *Y* and *Z* variables, or equivalently, by the corresponding values of H_1 and H_0 which are deduced from equations (1) and (2). A few remarks about this running example are in order.

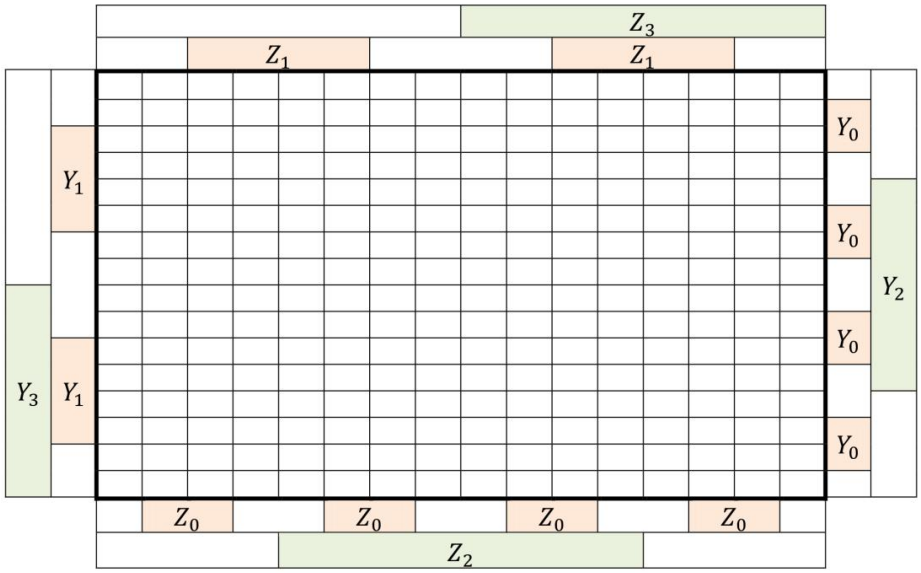
- Through this example is suggested for a context other than those typically employed in QCA applications, it nevertheless represents a truth table for a partially-defined (incompletely-specified) Boolean function, i.e., a function with logical remainders (in QCA jargon).
- The truth table is notorious for its extremely few specifications. Out of $2^8 = 256$ possible configurations, data is available for only 7 configurations. The fact is vividly illustrated by Figs. 2 or 3, in each of which the Karnaugh map has 256 cells, with only 7 of them of assigned or asserted entries of 0 or 1. The remaining 249 cells are left blank, indicating that they are logical remainders (in QCA terminology) or don't-cares (in the digital-design terminology).
- No consistency cutoff is assigned to the values asserted for f in Table 1. We assume a perfect consistency of 1.00 for each of the 7 configurations specified in Table 1.

Table 1. Truth table of the running example, adapted from Crama et al. [1]. The truth table expresses a single endogenous factor (outcome) in terms of eight exogenous factors (inputs) expressed as dichotomous (binary) variables that can be compressed into two hexadecimal variables H_1H_0. Only 7 out of 256 lines or configurations are specified.

Y_3	Y_2	Y_1	Y_0	Z_3	Z_2	Z_1	Z_0	H_1	H_0	f
0	1	0	1	0	1	1	0	5	6	1
1	1	0	1	1	0	0	1	D	9	1
0	1	1	0	1	0	0	1	6	9	1
1	0	1	0	1	0	1	0	A	A	0
0	0	0	1	1	1	0	0	1	C	0
1	1	0	1	0	1	0	1	D	5	0
0	0	1	0	1	0	1	0	2	A	0

Table 2. Hexadecimal 'digits' in terms of decimal digits and binary bits

Hexadecimal 'Digits'	Decimal Digits	Binary Bits
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111



f

f

Fig. 1. A regular and modular eight-variable Karnaugh map

3. CONSTRUCTION AND USE OF THE KARNAUGH MAP

Construction of the Karnaugh map is based on repeated application of the Boole-Shannon Expansion [44-46], which takes the following form when implemented *w.r.t.* a single variable X_{k}

$$f(\boldsymbol{X}) = (\bar{X}_k \wedge f(\boldsymbol{X}|0_{\mathrm{k}})) \vee (X_{\mathrm{k}} \wedge f(\mathbf{X}|1_{\mathrm{k}})) \quad (3)$$

This Boole-Shannon Expansion expresses the Boolean function $f(\boldsymbol{X})$ in terms of its two subfunctions $f(\boldsymbol{X}|0_{\mathrm{k}})$ and $f(\boldsymbol{X}|1_{\mathrm{k}})$. These subfunctions are equal to the Boolean quotients $f(\boldsymbol{X})/\bar{X}_k$ and $f(\boldsymbol{X})/X_{\mathrm{k}}$, and hence are obtained by restricting X_k in the expression $f(\boldsymbol{X})$ to 0 and 1, respectively. If $f(\boldsymbol{X})$ is an expression of n variables, the two subfunctions $f(\boldsymbol{X}|0_{\mathrm{k}})$ and $f(\boldsymbol{X}|1_{\mathrm{k}})$ are functions of at most $(n-1)$ variables. If the Boole-Shannon expansion is applied in sequence to the n variables of $f(\boldsymbol{X})$,

the expansion tree is a complete binary tree (usually called a Binary Decision Diagram) of 2^n leaves. These leaves are functions of no variables, or constants, and equal the entries of a corresponding conventional Karnaugh map of $f(\boldsymbol{X})$. Sibling nodes (nodes at the same level) of this expansion tree constitute the entries of a variable-entered (or a map-entered) Karnaugh map of $f(\boldsymbol{X})$ [17,45]. Various types of map folding allow the replacement of an n-variable map by two $(n-1)$ -variable maps [34,35,38,18,47-51]. Conversely, the n-variable map might be viewed as a *map-entered map* [52-55] with a new map variable, say Z_n, and two major cells, each of which having the size of an $(n-1)$-variable map. Such a view might be repeated recursively, so as to construct a map of any desirable size.

If a Karnaugh map is used to represent a Boolean function $f(\boldsymbol{X})$, then the map can be split into two halves (with respect to the borders of the variable X_{k}) representing the internal and external domains of this variable. These half maps depict, respectively, the two subfunctions or Boolean quotients $f(\boldsymbol{X})/\bar{X}_k$ and $f(\boldsymbol{X})/X_{\mathrm{k}}$, which are functions of the $(n-1)$ variables of $\boldsymbol{X}$ other than X_{k}. We say that $f(\boldsymbol{X})$ is vacuous in (independent of) X_{k} if the following relation is identically satisfied [34,49,51,56].

$$f(\boldsymbol{X})/\bar{X}_k = f(\boldsymbol{X})/X_{\mathrm{k}}. \tag{4}$$

According to (3), if (4) is identically satisfied then $f(\boldsymbol{X})$ becomes equal to each of $f(\boldsymbol{X})/\bar{X}_k$ and $f(\boldsymbol{X})/X_{\mathrm{k}}$, and hence it becomes a function of the $(n-1)$ variables of $\boldsymbol{X}$ other than X_{k}. This means that X_{k} is now guaranteed to be not a supporting variable of $f(\boldsymbol{X})$. The variable X_{k} must be included in a set of supporting variables of $f(\boldsymbol{X})$ if at least a cell in the half map $f(\boldsymbol{X})/\bar{X}_k$ is found to differ in its entry from its image cell in the half map $f(\boldsymbol{X})/X_{\mathrm{k}}$ (the cell obtained through reflection of the original cell with respect to the nearest border of X_{k}). Note that only the 0 and 1 entries are viewed as different, opposing or contradictory, while a don't-care entry (d) can be made to match either 0 or 1. The variable X_{k} might be excluded from a set of supporting variables of $f(\boldsymbol{X})$ if (a) no case of the afore-mentioned pair of contradictory mirror-image cells can be found, and (b) appropriate conditions are placed on don't-care cells in either the $f(\boldsymbol{X})/\bar{X}_k$ or $f(\boldsymbol{X})/X_{\mathrm{k}}$ half maps that are images of 0-entered or 1-entered cells in the other half so as to ensure that (4) is satisfied.

In this paper, we will use a regular and modular form of the Karnaugh map that appeared earlier in [41-43,57-69], and is such that

a) The rectangular shape of the map cell is retained.
b) The internal domain of the $(n+1)$ st variable is introduced to be centered around the boundary lines of the $(n-1)$st variable.

We note that the process outlined in (b) above can be, in theory, indefinitely continued. Hence, there is no theoretical upper bound on the size of the Karnaugh map constructed this way. However, as the number of variables increases, the size of the map increases exponentially, and its utility diminishes very quickly due to prohibitively increasing difficulty. As a demonstration of the usefulness of the aforementioned version of the Karnaugh map, we present its case of eight variables herein. This map (shown in Fig. 1) is exactly what we need for our current example.

4. KARNAUGH MAP SOLUTION OF THE RUNNING EXAMPLE

Fig. 1 presents the regular and modular eight-variable 256-cell Karnaugh map used herein. Fig. 2 indicates locations of specified configurations of the running example given in Sec.2. These locations are taken from Table 1, and are depicted here as two hexadecimal numbers, which are colored green for high assertion (1) and red for low assertion (0). As usual (in digital-design circles), the unspecified cells or configurations are left blank, and are implicitly understood to be entered with don't-cares (called logical remainders in QCA jargon). For convenience, the numerical values of the hexadecimal numbers are used to mark the vertical and horizontal axes of the map. Fig. 3 replaces the input variables $(Y_3, Y_2, Y_{,1} Y_0, Z_3, Z_2, Z_1, Z_0)$ in Fig. 1 by $(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)$ to restore the original names or uniform symbols in Crama et al. [1], and identify the cells of high assertion (1) and low assertion (0) by their entries rather than their locations.

At this stage, we discover that our present example is much more difficult and versatile than the ones we encountered earlier [41-43]. The present example has fewer specified configurations, and none of its variables is an essential supporting variable, i.e., its outcome

function can be made independent of any single variable. For example, Fig. 4 manages to make f independent of the variable X_2 by imposing appropriate conditions for f in Fig. 3, so that cells that are equidistant from the borders of X_2 (highlighted in red) are ensured to have equal entries. Fig. 5 pushes this issue further through discovery that the outcome f can be made independent not only of X_2 but also of each of X_4, X_6 and X_8. This causes the outcome f to become dependent on members of the set of the four remaining variables $\{X_1, X_3, X_5, X_7\}$, and hence proves that this set is a minimal set of supporting variables. In Fig. 6, the Karnaugh map of Fig. 5 is reduced to one of these four variables $\{X_1, X_3, X_5, X_7\}$. This map inherits the three asserted high (1) and four asserted low (0) configurations of the original map. With this minimal set of supporting variables, the function has six IDFs (in agreement with Crama et al. [1,Table 4]). There is a single essential prime implicant $X_3\,\overline{X_7}$ (highlighted in yellow) that covers the starred 1. The 1 in the cell $\overline{X_1}\;\overline{X_3}\;\overline{X_5}\;X_7$ (colored blue) might be covered by any of the three four-cell loops $\overline{X_1}\;\overline{X_5}$, $\overline{X_3}\;X_7$ or $\overline{X_5}\;X_7$, while the 1 in the cell $X_1\,\overline{X_3}\,X_5\,\overline{X_7}$ (colored pink) might be covered by any of the two two-cell loops $X_1\,X_5\,\overline{X_7}$ and $X_1\,\overline{X_3}\;X_5$.

Fig. 7 takes another turn to prove that the alternative set of four variables $\{X_1, X_4, X_5, X_7\}$ is yet another minimal set of supporting variables. This is achieved through discovery that the outcome f can be made independent of each of X_2, X_3, X_6 and X_8, and subsequent observation that this outcome becomes dependent on members of the afore-mentioned set. Fig. 8 reduces the Karnaugh map of Fig. 7 to one of the four variables $\{X_1, X_4, X_5, X_7\}$. This map again inherits the three asserted high (1) and four asserted low (0) configurations of the original map. With this minimal set of supporting variables, the function has another set of six IDFs (in agreement with Crama et al. [1,Table 4]). There is a single essential prime implicant $\overline{X_4}\;\overline{X_7}$ (highlighted in yellow) that covers the starred 1. The 1 in the cell $\overline{X_1}\;X_4\;\overline{X_5}\;X_7$ (colored blue) might be covered by any of the three four-cell loops $\overline{X_1}\;\overline{X_5}$, $X_4\,X_7$ or $\overline{X_5}\;X_7$, while the 1 in the cell $X_1\;X_4\,X_5\,\overline{X_7}$ (colored pink) might be covered by any of the two two-cell loops $X_1\,X_5\,\overline{X_7}$ or $X_1\,X_4\,X_5$.

Fig. 9 now proves that the set of three variables $\{X_1, X_2, X_5\}$ is a minimal set of supporting variables through discovery that f can be made independent of each of X_3, X_4, X_6, X_7 and X_8 but then it becomes solely dependent on (and minimally supported by) members of that set. The result in this figure is much more elegant than the ones in Figs. 5 and 7 thanks to the perfect visual adjacency of the reduced cells and to the decreased cardinality of the supporting set. Fig. 10 reduces the Karnaugh map of Fig. 9 to one of the three variables $\{X_1, X_2, X_5\}$. This map inherits the three asserted high (1) and the four asserted low (0) configurations of the original map (with two of the 0's combined). With this minimal set of supporting variables, the function has two IDFs (in agreement with Crama et al. [1,Table 4]). There is a single essential prime implicant $X_2\,X_5$ (highlighted in yellow) that covers the starred 1, and also covers the 1 in the cell above, thereby turning it into a don't care (since, for further processing, this cell can (but does not have to) be covered). The 1 in the cell $\overline{X_1}\;X_2\;\overline{X_5}$ (colored blue) might be covered by any of the two two-cell loops $\overline{X_1}\;X_2$ or $\overline{X_1}\;\overline{X_5}$.

With three minimal sets of supporting variables so far discovered, Fig. 11 now adds a fourth one. This figure proves that the set of three variables $\{X_2, X_6, X_8\}$ is such a fourth minimal set of supporting variables through discovery that f can be made independent of each of X_1, X_3, X_4, X_5 and X_7 but then it becomes minimally supported by members of that set. Fig. 12 reduces the Karnaugh map of Fig. 9 to one of the three variables $\{X_2, X_6, X_8\}$. This map inherits the three asserted high (1) configurations (with two of the 1's combined), and the four asserted low (0) configurations of the original map (with two of the 0's combined). With this minimal set of supporting variables, the function has two IDFs (in agreement with Crama et al. [1,Table 4]). There is a single essential prime implicant $X_2\,\overline{X_8}$ (highlighted in yellow) that covers the starred 1. The 1 in the cell $X_2\,\overline{X_6}\;X_8$ (colored blue) might be covered by any of the two two-cell loops $X_2\,\overline{X_6}$ or $\overline{X_6}\;X_8$.

Now, Fig. 13 acts as a proof that the set of three variables $\{X_2, X_5, X_7\}$ is a minimal set of supporting variables through discovery that f can be made independent of each of X_1, X_3, X_4, X_6, which makes it minimally supported by members of that set. Figure 14 reduces the Karnaugh map of Fig. 13 to one of the three variables $\{X_2, X_5, X_7\}$. This map inherits the three asserted high (1) configurations (with two of the 1's combined), and the four asserted low (0) configurations of the original map (with two of the 0's combined). With this minimal set of

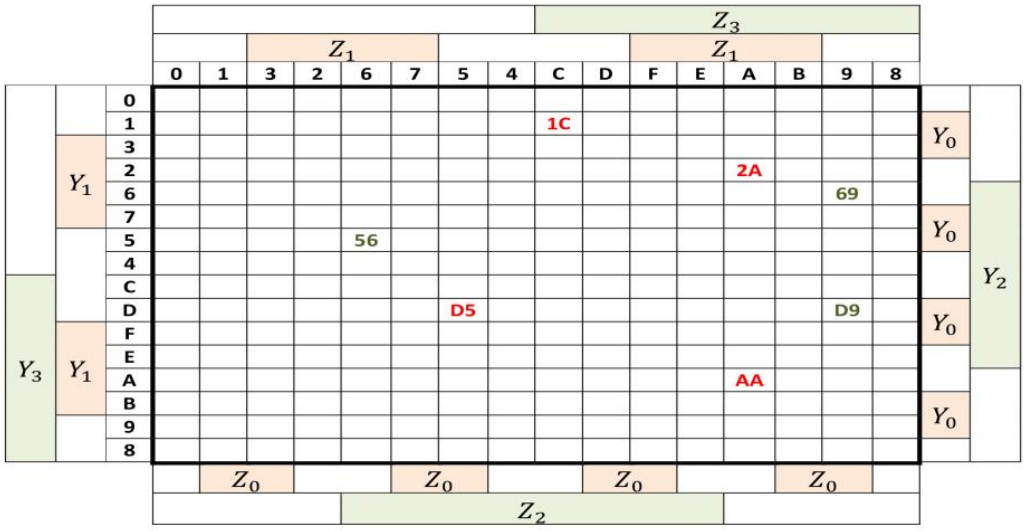
supporting variables, the function has two more IDFs (in agreement with Crama et al. [1,Table 4]). There is a single essential prime implicant $X_2\ X_5$ (highlighted in yellow) that covers the starred 1. The 1 in the cell $X_2\ \overline{X_5}\ X_7$ (colored blue) might be covered by any of the two two-cell loops $X_2\ X_7$ or $\overline{X_5}\ X_7$.

Similarly to the few previous figures, Fig. 15 adds a proof that the set of three variables $\{X_1, X_2, X_6\}$ is also a minimal set of supporting variables through discovery that the outcome f can be made independent of each of X_3, X_4, X_5, X_7 and X_8, followed by asserting that this outcome then becomes minimally supported by members of that set. Fig. 16 reduces the Karnaugh map of Fig. 15 to one of the three variables $\{X_1, X_2, X_6\}$. This map inherits the three asserted high (1) and the four asserted low (0) configurations of the original map. With this minimal set of supporting variables, the function has a single IDF (in agreement with Crama et al. [1,Table 4]). There is an essential prime implicant $X_2\ \overline{X_6}$ (highlighted in yellow) that covers the single-starred 1, and another essential prime implicant $\overline{X_1}\ X_2$ that covers the double-starred one. The function has a single don't-care cell, which is assigned 0 for the solution above. Under this assignment, the complement of the function also has a single IDF. The situation in Figs. 15 and 16 is the most appealing so far according to certain arguments [22,41-43,62], but its appealing or desirable nature is no excuse to let it dominate or rule out other possibilities.

Now, Fig. 17 proves that the set of two variables $\{X_5, X_8\}$ is a minimal set of supporting variables through discovery that f can be made independent of each of X_1, X_2, X_3, X_4, X_6 and X_7 but then it becomes minimally supported by members of that set. Fig. 18 reduces the Karnaugh map of Fig. 17 to one of the two variables $\{X_5, X_8\}$. This map inherits the three asserted high (1) configurations (with two of the 1's combined), and the four asserted low (0) configurations of the original map (with three of the 0's combined). With this minimal set of supporting variables, the function is completely specified and has a single IDF (in agreement with Crama et al. [1,Table 4]). There is an essential prime implicant $X_5\ X_8$ (highlighted in yellow) that covers the starred 1, and another essential prime implicant $\overline{X_5}\ \overline{X_8}$ (highlighted in green) that covers the double-starred 1. The complement of the function also has a single IDF. The situation in Figs. 17 and 18 is even more appealing than the ones in previous figures, but again, this is no reason for the current solution to dominate earlier ones in QCA sense or context.

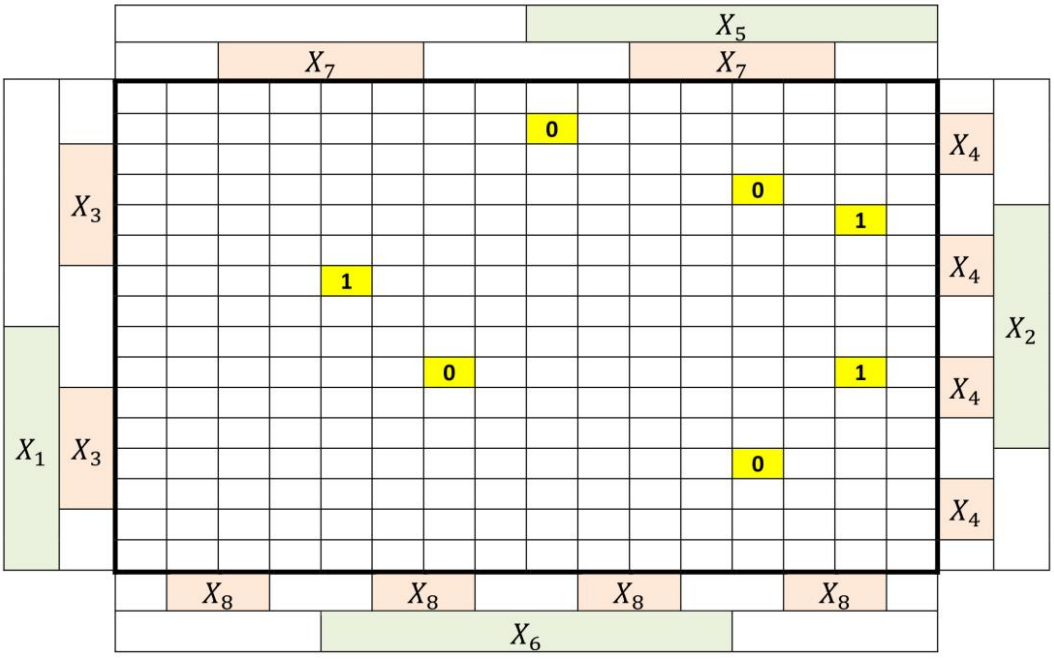
Finally, Fig. 19 presents a proof that the set of two variables $\{X_6, X_7\}$ is a minimal set of supporting variables through discovery that f can be made independent of each of X_1, X_2, X_3, X_4, X_5 and X_8 but then it becomes minimally supported by members of that set. Fig. 20 reduces the Karnaugh map of Fig. 19 to one of the two variables $\{X_6, X_7\}$. This map inherits the three asserted high (1) configurations (with two of the 1's combined), and the four asserted low (0) configurations of the original map (with two of the 0's combined, and the other two 0's combined). With this minimal set of supporting variables, the function is completely specified and has a single IDF (in agreement with Crama et al. [1,Table 4]). There is an essential prime implicant $X_6\ X_7$ (highlighted in yellow) that covers the starred 1, and another essential prime implicant $\overline{X_6}\ \overline{X_7}$ (highlighted in green) that covers the double-starred 1. The situation in Figs. 19 and 20 resembles that in Figs. 17 and 18.

Now, we have completely recovered all the minimal sets of supporting sets obtained via integer linear programming, together with their associated sets of IDFs [1,Table 4]. Admittedly, we have been guided by our pre-knowledge of Table 4 in [1]. However, our heuristic procedure would work even without such pre-knowledge. Fig. 21 is an attempt to prove that the set of two variables $\{X_1, X_2\}$ is a minimal set of supporting variables, which is doomed to failure. The figure shows that f can (for example) be made independent of each of X_3, X_4, X_6, X_7 and X_8 but then it becomes impossible to render it independent of X_5 because the two half maps across the X_5 borders (highlighted in red) cannot be matched. On the other hand, Fig. 22 attempts to prove that the set of four variables $\{X_1, X_2, X_5, X_6\}$ is a minimal set of supporting variables. The function f can be made independent of each of X_3, X_4, X_7 and X_8 as shown but the reduced map of f in terms of X_1, X_2, X_5 and X_6 does not exhibit minimality of the set of supporting variables. In fact, Fig. 23 illustrates that the set of four supporting variables $\{X_1, X_2, X_5, X_6\}$ is not a minimal one, as it can drop either X_6 or X_5, thereby being reduced to either the set $\{X_1, X_2, X_5\}$ (same as the one in Fig. 10) or the set $\{X_1, X_2, X_6\}$ (same as the one in Fig. 16). These two latter sets are minimal as each of them cannot drop further any variable.



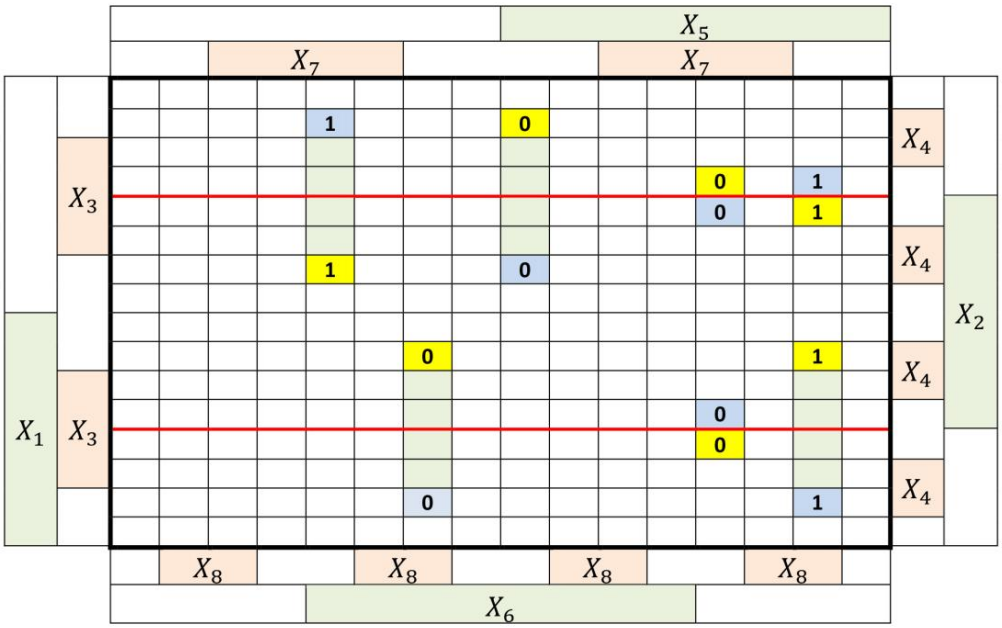
f

Fig. 2. Locations of specified configurations of the running example, depicted as two hexadecimal numbers, and colored green for high assertion (1) and red for low assertion (0). For convenience, the numerical values of the hexadecimal numbers are shown



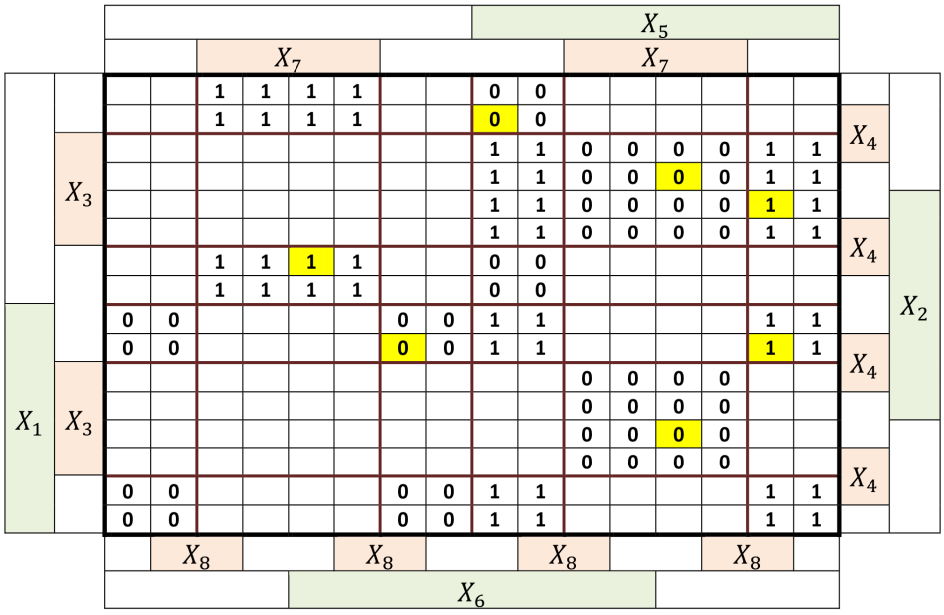
f

Fig. 3. Replacing the input variables $(Y_3, Y_2, Y_1, Y_0, Z_3, Z_2, Z_1, Z_0)$ in Fig. 1 by $(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)$ to restore the original names in Crama et al. [1], and identifying the cells of high assertion (1) and low assertion (0)



f

Fig. 4. Imposing conditions for f in Fig. 3 to make it independent of X_2. Cells that are equidistant from the borders of X_2 (highlighted in red) are ensured to have equal entries



f

Fig. 5. Proof that the set of four variables $\{X_1, X_3, X_5, X_7\}$ is a minimal set of supporting variables through discovery that f can be made independent of each of X_2, X_4, X_6 and X_8 but then it becomes dependent on members of that set

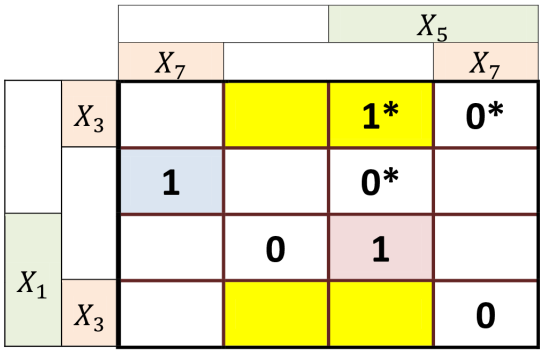


Fig. 6. The Karnaugh map of Fig. 5 reduced to one of the four variables $\{X_1, X_3, X_5, X_7\}$. This map inherits the three asserted high (1) and four asserted low (0) configurations of the original map. With this minimal set of supporting variables, the function has six IDFs (in agreement with Crama et al. [1, Table 4]). There is a single essential prime implicant $X_3\,\overline{X_7}$ (highlighted in yellow) that covers the starred 1. The 1 in the cell $\overline{X_1}\;\overline{X_3}\;\overline{X_5}\;X_7$ (colored green) might be covered by any of the three four-cell loops $\overline{X_1}\;\overline{X_5}$, $\overline{X_3}\;X_7$ or $\overline{X_5}\;X_7$, while the 1 in the cell $X_1\;\overline{X_3}\;X_5\;\overline{X_7}$ (colored pink) might be covered by any of the two two-cell loops $X_1\;X_5\;\overline{X_7}$ and $X_1\;\overline{X_3}\;X_5$

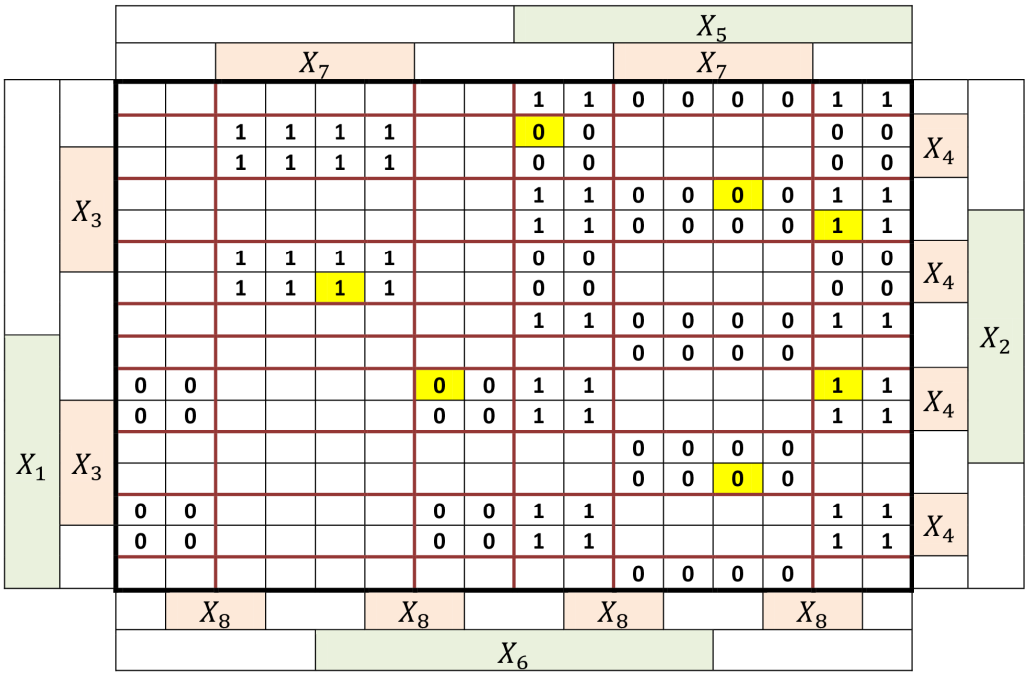


Fig. 7. Proof that the set of four variables $\{X_1, X_4, X_5, X_7\}$ is a minimal set of supporting variables through discovery that f can be made independent of each of X_2, X_3, X_6 and X_8 but then it becomes dependent on members of that set

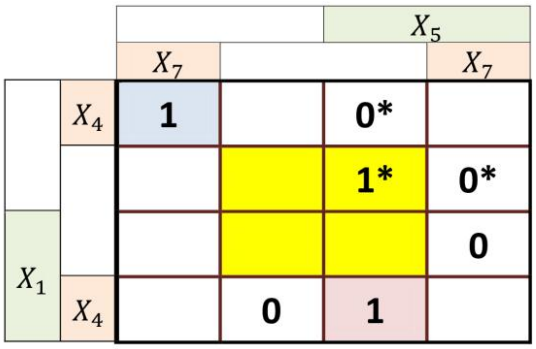


Fig. 8. The Karnaugh map of Fig. 7 reduced to one of the four variables $\{X_1, X_4, X_5, X_7\}$. This map again inherits the three asserted high (1) and four asserted low (0) configurations of the original map. With this minimal set of supporting variables, the function has another set of six IDFs (in agreement with Crama et al. [1, Table 4]). There is a single essential prime implicant $\overline{X_4}\ \overline{X_7}$ (highlighted in yellow) that covers the starred 1. The 1 in the cell $\overline{X_1}\ X_4\ \overline{X_5}\ X_7$ (colored blue) might be covered by any of the three four-cell loops $\overline{X_1}\ \overline{X_5}$, $X_4\ X_7$ or $\overline{X_5}\ X_7$, while the 1 in the cell $X_1\ X_4\ X_5\ \overline{X_7}$ (colored pink) might be covered by any of the two two-cell loops $X_1\ X_5\ \overline{X_7}$ or $X_1\ X_4\ X_5$

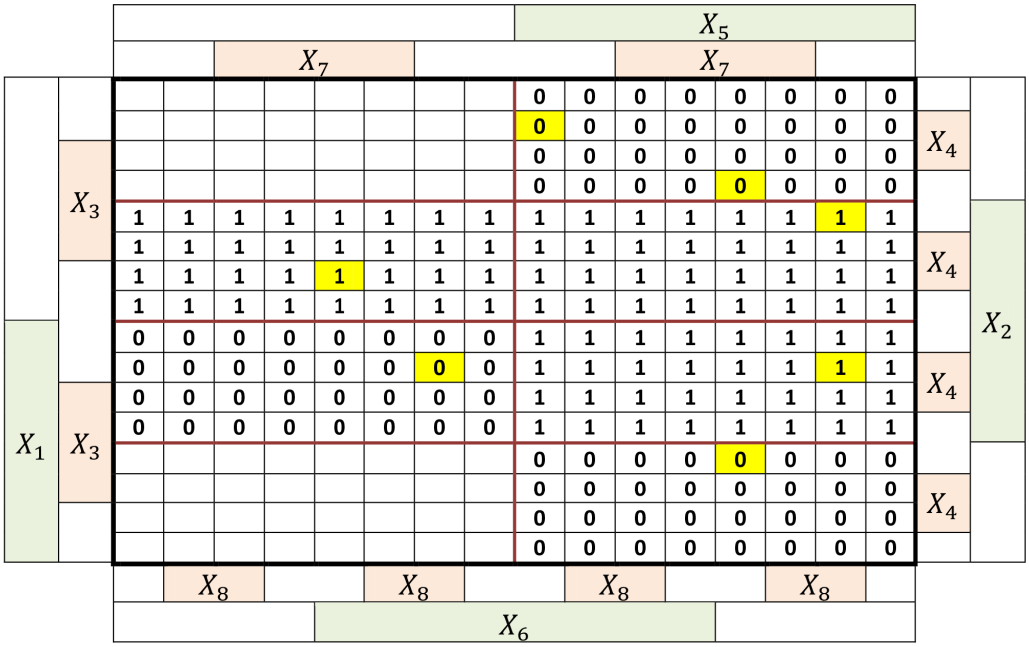


Fig. 9. Proof that the set of three variables $\{X_1, X_2, X_5\}$ is a minimal set of supporting variables through discovery that f can be made independent of each of X_3, X_4, X_6, X_7 and X_8 but then it becomes solely dependent on (and minimally supported by) members of that set. The result in this figure is much more elegant than the ones in Figs. 5 and 7 thanks to the perfect visual adjacency of the reduced cells and to the decreased cardinality of the supporting set

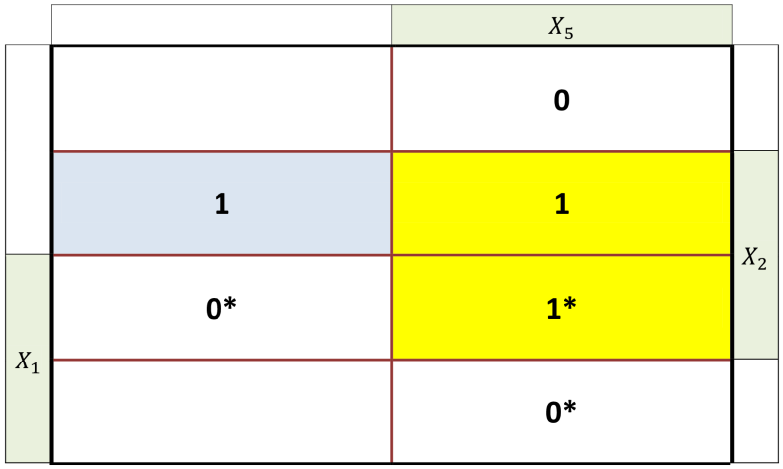


Fig. 10. The Karnaugh map of Fig. 9 reduced to one of the three variables $\{X_1, X_2, X_5\}$. This map inherits the three asserted high (1) and the four asserted low (0) configurations of the original map (with two of the 0's combined). With this minimal set of supporting variables, the function has two IDFs (in agreement with Crama et al. [1, Table 4]). There is a single essential prime implicant $X_2\ X_5$ (highlighted in yellow) that covers the starred 1, and also covers the 1 in the cell above, thereby turning it into a don't care . The 1 in the cell $\overline{X_1}\ X_2\ \overline{X_5}$ (colored blue) might be covered by any of the two two-cell loops $\overline{X_1}\ X_2$ or $\overline{X_1}\ \overline{X_5}$

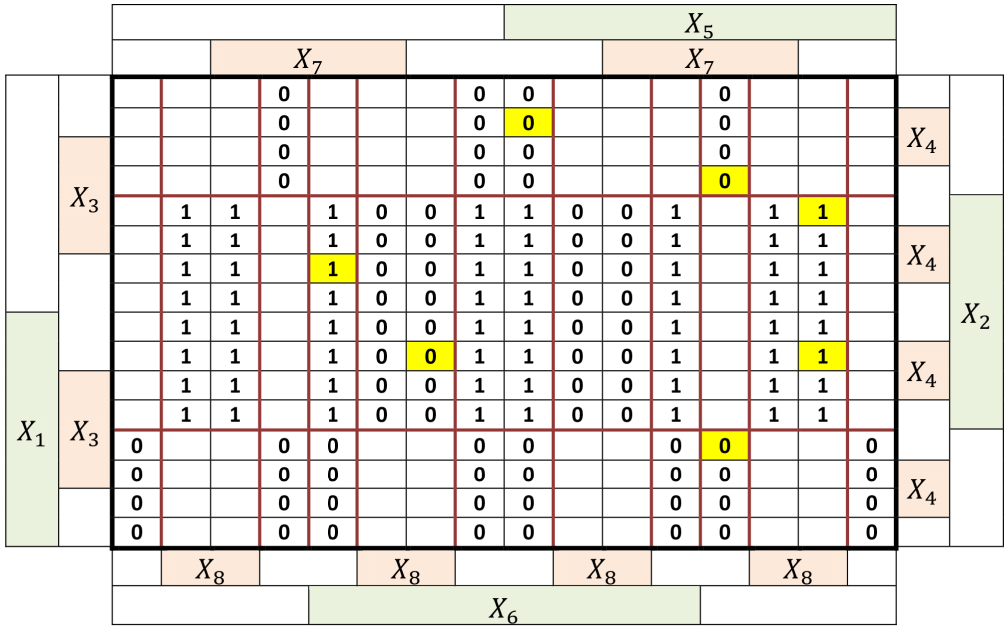
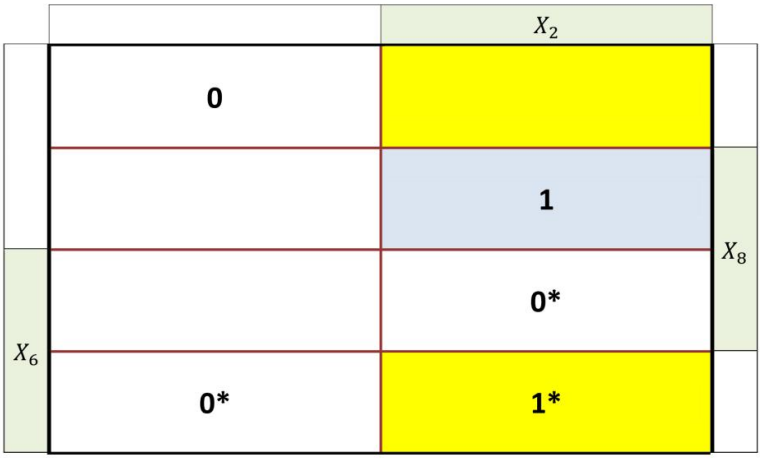
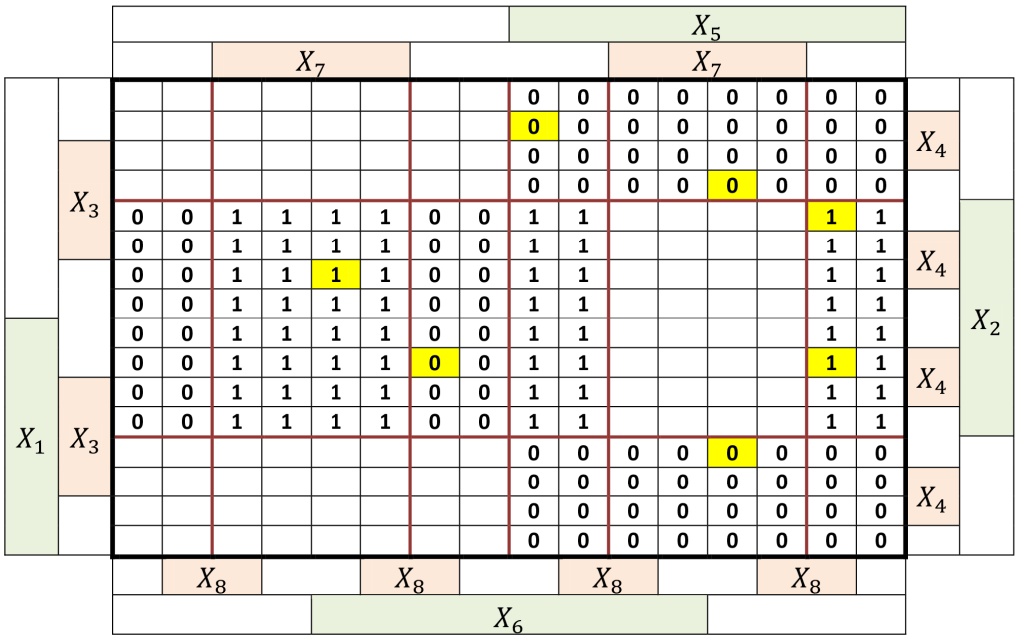


Fig. 11. Proof that the set of three variables $\{X_2, X_6, X_8\}$ is a minimal set of supporting variables through discovery that f can be made independent of each of X_1, X_3, X_4, X_5 and X_7 but then it becomes minimally supported by members of that set



f

Fig. 12. The Karnaugh map of Fig. 9 reduced to one of the three variables $\{X_2, X_6, X_8\}$. This map inherits the three asserted high (1) configurations (with two of the 1's combined), and the four asserted low (0) configurations of the original map (with two of the 0's combined). With this minimal set of supporting variables, the function has two IDFs (in agreement with Crama et al. [1, Table 4]). There is a single essential prime implicant $X_2\,\overline{X_8}$ (highlighted in yellow) that covers the starred 1. The 1 in the cell $X_2\,\overline{X_6}\,X_8$ (colored blue) might be covered by any of the two two-cell loops $X_2\,\overline{X_6}$ or $\overline{X_6}\,X_8$



f

Fig. 13. Proof that the set of three variables $\{X_2, X_5, X_7\}$ is a minimal set of supporting variables through discovery that f can be made independent of each of X_1, X_3, X_4, X_6 and X_8 but then it becomes minimally supported by members of that set

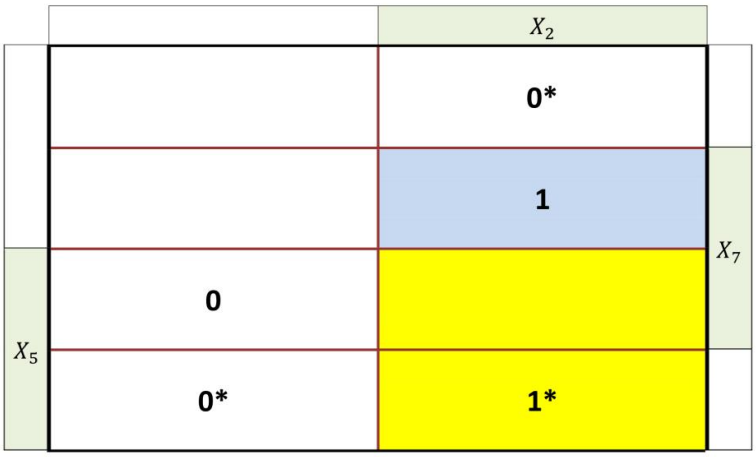
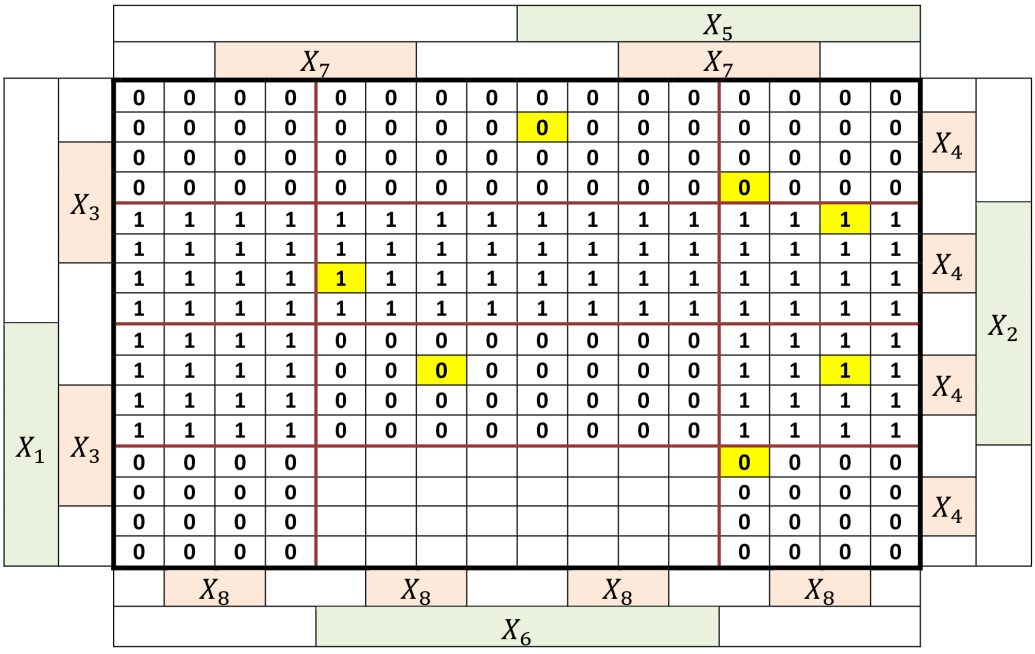


Fig. 14. The Karnaugh map of Fig. 13 reduced to one of the three variables $\{X_2, X_5, X_7\}$. This map inherits the three asserted high (1) configurations (with two of the 1's combined), and the four asserted low (0) configurations of the original map (with two of the 0's combined). With this minimal set of supporting variables, the function has two more IDFs (in agreement with Crama et al. [1, Table 4]). There is a single essential prime implicant $X_2\ X_5$ (highlighted in yellow) that covers the starred 1. The 1 in the cell $X_2\ \overline{X_5}\ X_7$ (colored blue) might be covered by any of the two two-cell loops $X_2\ X_7$ or $\overline{X_5}\ X_7$



f

Fig. 15. Proof that the set of three variables $\{X_1, X_2, X_6\}$ is a minimal set of supporting variables through discovery that f can be made independent of each of X_3, X_4, X_5, X_7 and X_8 but then it becomes minimally supported by members of that set

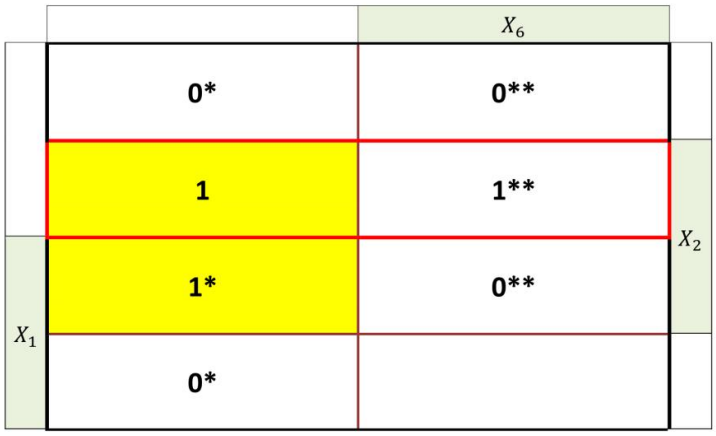


Fig. 16. The Karnaugh map of Fig. 15 reduced to one of the three variables $\{X_1, X_2, X_6\}$. This map inherits the three asserted high (1) and the four asserted low (0) configurations of the original map. With this minimal set of supporting variables, the function has a single IDF (in agreement with Crama et al. [1, Table 4]). There is an essential prime implicant $X_2\ \overline{X_6}$ (highlighted in yellow) that covers the single-starred 1, and another essential prime implicant $\overline{X_1}\ X_2$ that covers the double-starred one. The function has a single don't-care cell, which is assigned 0 for the solution above. Under this assignment, the complement of the function also has a single IDF

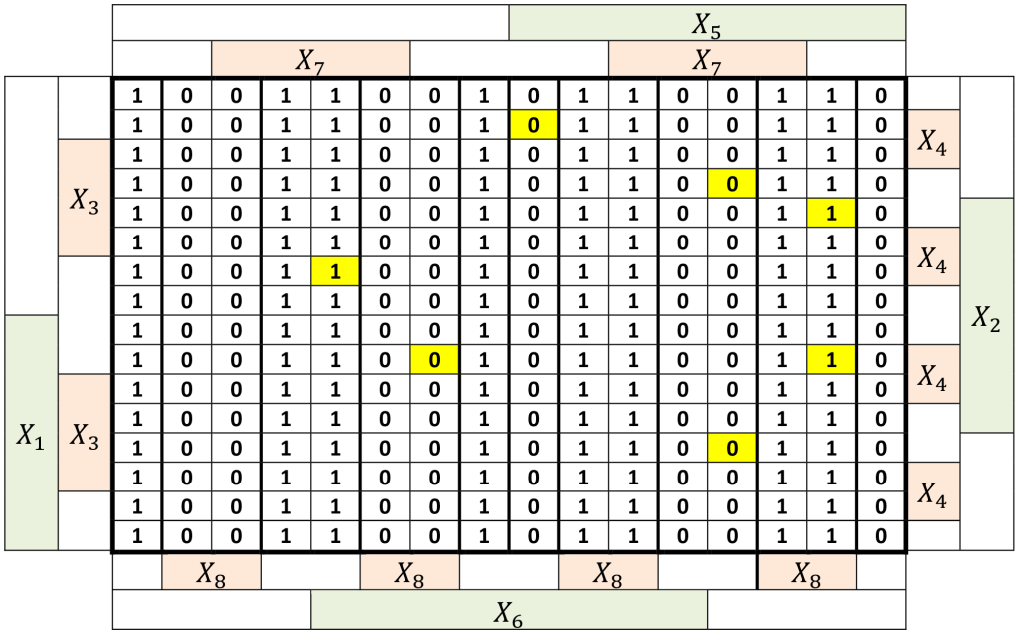


Fig. 17. Proof that the set of two variables $\{X_5, X_8\}$ is a minimal set of supporting variables through discovery that f can be made independent of each of X_1, X_2, X_3, X_4, X_6 and X_7 but then it becomes minimally supported by members of that set

		X_5
	1**	0
X_8	0	1*

f

Fig. 18. The Karnaugh map of Fig. 17 reduced to one of the two variables $\{X_5, X_8\}$. This map inherits the three asserted high (1) configurations (with two of the 1's combined), and the four asserted low (0) configurations of the original map (with three of the 0's combined). With this minimal set of supporting variables, the function is completely specified and has a single IDF (in agreement with Crama et al. [1, Table 4]). There is an essential prime implicant $X_5\, X_8$ (highlighted in yellow) that covers the starred 1, and another essential prime implicant $\overline{X_5}\ \overline{X_8}$ (highlighted in green) that covers the double-starred 1. The complement of the function also has a single IDF

							X_5										
			X_7								X_7						
		1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1	
		1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1	
	X_3	1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1	X_4
		1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1	
		1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1	
		1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1	X_4
		1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1	
		1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1	
		1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1	X_4
		1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1	
		1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1	
X_1	X_3	1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1	X_4
		1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1	
		1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1	

X_2

	X_8		X_8		X_8		X_8	
		X_6						

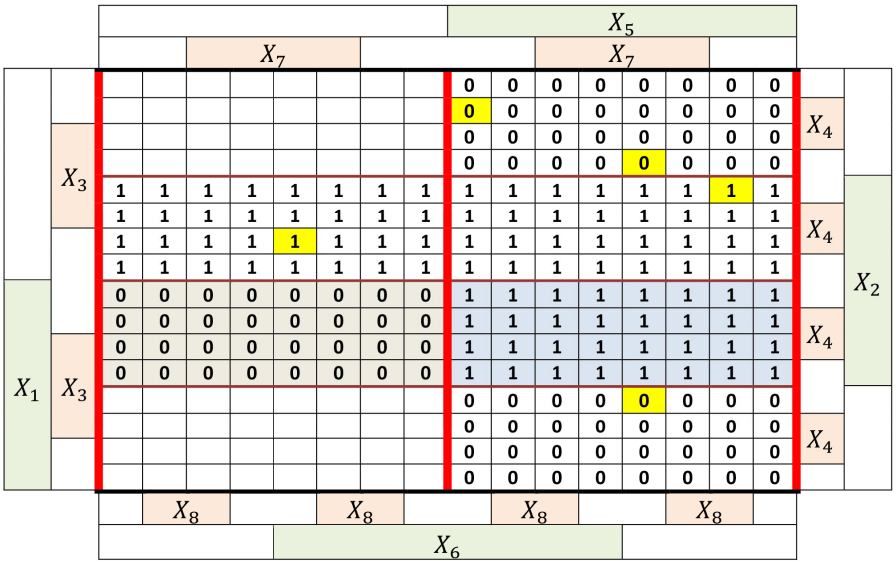
f

Fig. 19. Proof that the set of two variables $\{X_6, X_7\}$ is a minimal set of supporting variables through discovery that f can be made independent of each of X_1, X_2, X_3, X_4, X_5 and X_8 but then it becomes minimally supported by members of that set

		X_6
	1**	0
X_7	0	1*

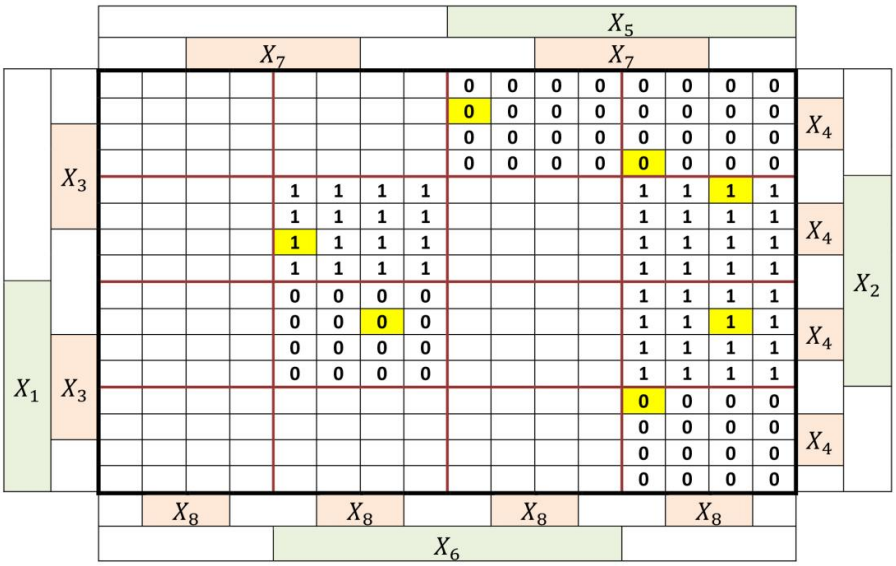
f

Fig. 20. The Karnaugh map of Fig. 19 reduced to one of the two variables $\{X_6, X_7\}$. This map inherits the three asserted high (1) configurations (with two of the 1's combined), and the four asserted low (0) configurations of the original map (with two of the 0's combined, and the other two 0's combined). With this minimal set of supporting variables, the function is completely specified and has a single IDF (in agreement with Crama et al. [1, Table 4]). There is an essential prime implicant $X_6\, X_7$ (highlighted in yellow) that covers the starred 1, and another essential prime implicant $\overline{X_6}\ \overline{X_7}$ (highlighted in green) that covers the double-starred 1



f

Fig. 21. An attempt to prove that the set of two variables $\{X_1, X_2\}$ is a minimal set of supporting variables is doomed to failure. The figure shows that f can be made independent of each of X_3, X_4, X_6, X_7 and X_8 but then it becomes impossible to render it independent of X_5 because the two half maps across the X_5 borders (highlighted in red) cannot be matched



f

Fig. 22. Attempt to prove that the set of four variables $\{X_1, X_2, X_5, X_6\}$ is a minimal set of supporting variables. The function f can be made independent of each of X_3, X_4, X_7 and X_8 as shown but the reduced map of f in terms of X_1, X_2, X_5 and X_6 does not exhibit minimality of the set of supporting variables

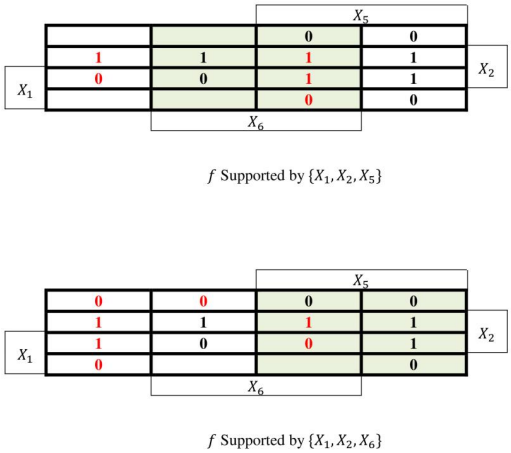


Fig. 23. Illustration that the set of four supporting variables $\{X_1, X_2, X_5, X_6\}$ is not a minimal one, as it can drop either X_6 or X_5, thereby being reduced to either the set $\{X_1, X_2, X_5\}$ (same as in Figure 10) or the set $\{X_1, X_2, X_6\}$ (same as in Fig. 16). These two later sets are minimal as each of them cannot drop further any variable

5. CONCLUSIONS

We used a regular and modular eight-variable Karnaugh map to explore a large problem of cause-effect relations that mimics problems of the newly-established discipline of Qualitative Comparative Analysis (QCA). This problem involves an eight-variable partially-defined Boolean function (PDBF), that is dominantly unspecified. Without using the integer-programming approach, we devised a simple map procedure to discover minimal sets of supporting variables in the outset before proceeding to seek IDF representations. According to our scheme of first identifying minimal sets of supporting variables, we avoided the task of drawing prime-implicant loops on the initial eight-variable map, and postponed this task till the map was dramatically reduced in size.

The procedure outlined herein has important consequences for Qualitative Comparative Analysis (QCA). These consequences are not expected to be welcomed by the QCA community, since they clearly indicate that the too-often strong results claimed by QCA adherents need to be checked and scrutinized. While the problem studied herein yielded twenty three IDF solutions distributed under umbrellas of eight minimal sets of supporting variables, mainstream QCA would simply use a program of Boolean minimization that produces a single minimal sum selected from the aforementioned set of twenty three IDFs. That minimal sum would have been a wonderful solution, had it been appropriately justified. In our opinion, more observations have to be made in order to narrow down the possibilities and decrease the number of candidate IDFs. Otherwise, our attempt to give preference to a certain IDF over the others should be context-specific rather than tool-specific.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

REFERENCES

1. Crama Y, Hammer PL, Ibaraki T. Cause-effect relationships and partially defined Boolean functions. Annals of Operations Research. 1988;16(1):299-325.
2. Boros E, Hammer PL, Ibaraki T, Kogan A. Logical analysis of numerical data. Mathematical programming. 1997;79(1):163-190.
3. Boros E, Hammer PL, Ibaraki T, Kogan A, Mayoraz E, Muchnik I. An implementation of logical analysis of data. IEEE Transactions on Knowledge and Data Engineering. 2000;12(2):292-306.
4. Alexe S, Blackstone E, Hammer PL, Ishwaran H, Lauer MS, Snader CE. Coronary risk prediction by logical analysis of data. Annals of Operations Research. 2003;119(1):15-42.

5. Hammer PL, Bonates TO. Logical analysis of data—An overview: From combinatorial optimization to medical applications. Annals of Operations Research. 2006;148(1):203-225.
6. Bonates TO, Hammer PL, Kogan A. Maximum patterns in datasets. Discrete Applied Mathematics. 2008;156(6):846-861.
7. Alexe G, Alexe S, Hammer PL, Kogan A. Comprehensive vs. comprehensible classifiers in logical analysis of data. Discrete Applied Mathematics. 2008;156(6):870-882.
8. Cutler RB, Muroga S. Useless prime implicants of incompletely specified multiple-output switching functions. International Journal of Computer and Information Sciences. 1980;9(4):337-350.
9. Falkowski BJ, Schafer I, Perkowski MA. Effective computer methods for the calculation of Rademacher-Walsh spectrum for completely and incompletely specified Boolean functions. IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems. 1992;11(10):1207-1226.
10. Boros E, Gurvich V, Hammer PL, Ibaraki T, Kogan A. Decomposability of partially defined Boolean functions. Discrete Applied Mathematics. 1995;62(1-3):51-75.
11. Makino K, Yano K, Ibaraki T. Positive and Horn decomposability of partially defined Boolean functions. Discrete Applied Mathematics. 1997;74(3):251-74.
12. Rushdi AMA. Karnaugh map, Encyclopedia of Mathematics. M. Hazewinkel (Editor), Boston, Kluwer Academic Publishers. 1997;I(Supplement):327-328. Available: http://eom.springer.de/K/k110040.html
13. Boros E, Ibaraki T, Makino K. Error-free and best-fit extensions of partially defined Boolean functions. Information and Computation. 1998;140(2):254-283.
14. Makino K, Hatanaka KI, Ibaraki T. Horn extensions of a partially defined Boolean function. SIAM Journal on Computing. 1999;28(6):2168-2186.
15. Makino K, Ibaraki T. Inner-core and outer-core functions of partially defined Boolean functions. Discrete Applied Mathematics. 1999;96:443-60.
16. Boros E, Ibaraki T, Makino K. Fully Consistent Extensions of Partially Defined Boolean Functions with Missing Bits. In IFIP International Conference on Theoretical Computer Science Springer, Berlin, Heidelberg. 2000;257-272.
17. Rushdi AM, Al-Yahya HA. Further improved variable-entered Karnaugh map procedures for obtaining the irredundant forms of an incompletely-specified switching function. Journal of King Abdulaziz University: Engineering Sciences. 2001;13(1):111-152.
18. Rushdi AM. Prime-implicant extraction with the aid of the variable-entered Karnaugh map. Umm Al-Qura University Journal: Science, Medicine and Engineering. 2001;13(1):53-74.
19. Boros E, Ibaraki T, Makino K. Variations on extending partially defined Boolean functions with missing bits. Information and Computation. 2003;180(1):53-70.
20. Crama Y, Hammer PL. Boolean Functions: Theory, Algorithms, and Applications. Cambridge University Press; 2011.
21. Rushdi AM, Albarakati HM. Prominent classes of the most general subsumptive solutions of Boolean equations. Information Sciences. 2014;281:53-65.
22. Rushdi AMA, Badawi RMS. Computer engineers look at qualitative comparative analysis. International Journal of Mathematical, Engineering and Management Sciences (IJMEMS). 2019;4(4):851-860.
23. Valiant LG. A theory of the learnable. Communications of the ACM. 1984;27(11):1134-1142.
24. Kearns M, Li M, Pitt L, Valiant L. On the learnability of Boolean formulae. In Proceedings of the Nineteenth Annual ACM Symposium on Theory of Computing. 1987;285-295.
25. Natarajan BK. On learning Boolean functions. In Proceedings of the Nineteenth Annual ACM Symposium on Theory of Computing. 1987;296-304.
26. Rivest RL. Learning decision lists. Machine learning. 1987;2(3):229-246.
27. Pitt L, Valiant LG. Computational limitations on learning from examples. Journal of the ACM (JACM). 1988;35(4):965-984.
28. Bagallo G, Haussler D. Boolean feature discovery in empirical learning. Machine learning. 1990;5(1):71-99.
29. Kearns M, Valiant L. Cryptographic limitations on learning Boolean formulae

and finite automata. Journal of the ACM (JACM). 1994;41(1):67-95.
30. Hammer PL, Kogan A, Rothblum UG. Evaluation, strength, and relevance of variables of Boolean functions. SIAM Journal on Discrete Mathematics. 2000;13(3):302-312.
31. Eiter T, Ibaraki T, Makino K. Decision lists and related Boolean functions. Theoretical Computer Science. 2002;270(1-2):493-524.
32. Couceiro M, Hug N, Prade H, Richard G. Analogy-preserving functions: a way to extend Boolean samples. In IJCAI 2017-26th International Joint Conference on Artificial Intelligence. 2017;1-7.
33. Couceiro M, Hug N, Prade H, Richard G. Behavior of analogical inference wrt Boolean functions. In IJCAI 2018-27th International Joint Conference on Artificial Intelligence 2018;2057-2063.
34. Rushdi AMA, Badawi RMS. Karnaugh-map utilization in Boolean analysis: The case of war termination. Journal of Qassim University: Engineering and Computer Sciences. 2017;10(1):53-88.
35. Rushdi AMA, Badawi RS. Karnaugh map utilization in Coincidence Analysis, Journal of King Abdulaziz University: Faculty of Computers and Information Technology Sciences. 2017;6(1-2):37-44.
36. Baumgartner M, Thiem A. Model ambiguities in configurational comparative research. Sociological Methods & Research. 2017;46(4):954-987.
37. Rushdi AMA. Utilization of Karnaugh maps in multi-value qualitative comparative analysis, International Journal of Mathematical, Engineering and Management Sciences (IJMEMS). 2018;3(1):28-46.
38. Rushdi RA, Rushdi AM. Karnaugh-map utility in medical studies: The case of Fetal Malnutrition. International Journal of Mathematical, Engineering and Management Sciences (IJMEMS). 2018;3(3): 220-244.
39. Thiem A, Mkrtchyan L, Haesebrouck T, Sanchez D. Algorithmic bias in social research: A meta-analysis. PloS one. 2020 Jun 8;15(6):e0233625.
40. Baumgartner M, Thiem A. Often trusted but never (properly) tested: evaluating qualitative comparative analysis. Sociological Methods & Research. 2020;49(2):279-311.
41. Rushdi AMA, Badawi RS. Utilization of eight-variable Karnaugh maps in the exploration of problems of Qualitative Comparative Analysis. Asian Journal of Research in Computer Science. 2021;8(2):57-84.
42. Rushdi AMA, Badawi RS. Utilization of eight-variable Karnaugh maps in the exploration of problems of Qualitative Comparative Analysis. Asian Journal of Education and Social Studies. 2021;17(2):26-42.
43. Rushdi AMA, Badawi RS. Handling partially-defined Boolean functions in Qualitative Comparative Analysis by deriving the minimal sets of supporting variables before identifying the irredundant sums, Journal of King Abdulaziz University: Faculty of Computers and Information Technology Sciences. 2021;10(2):1-44.
44. Brown FM. Boolean reasoning: The logic of Boolean equations. Kluwer Academic Publishers, Boston, USA; 1990.
45. Rushdi AMA, Ghaleb FAM. A tutorial exposition of semi-tensor products of matrices with a stress on their representation of Boolean functions. Journal of King Abdulaziz University: Computers and Information Technology Sciences. 2016;5(1-2):3-30.
46. Rushdi AM, Rushdi MA. Switching-algebraic analysis of system reliability, Chapter 6 in M. Ram and P. Davim (Editors), Advances in Reliability and System Engineering, Management and Industrial Engineering Series, Springer International Publishing, Cham, Switzerland. 2017;139-161.
47. Rushdi AM. Map differentiation of switching functions. Microelectronics and Reliability, 1986;26(5):891-907.
48. Rushdi AM, Al-Yahya HA. Derivation of the complete sum of a switching function with the aid of the variable-entered Karnaugh map. Journal of King Saud University: Engineering Sciences. 2001;13(2): 239-268.
49. Rushdi AMA, Ba-Rukab OM. Calculation of Banzhaf voting indices utilizing variable-entered Karnaugh maps. British Journal Mathematics and Computer Science. 2017;20(4):1-17.
50. Rushdi AM, Rushdi MA. Mathematics and examples of the modern syllogistic method of propositional logic. In Ram, M. (Editor), Mathematics Applied in Information

Systems, Bentham Science Publishers, Emirate of Sharjah, United Arab Emirates. 2018;Chapter 6:123-167.
51. Rushdi AM, Ba-Rukab OM. Translation of Weighted Voting Concepts to the Boolean Domain: The Case of the Banzhaf Index, Chapter 10 in Advances in Mathematics and Computer Science Vol. 2, Book Publisher International, Hooghly, West Bengal, India, 2020:122-140.
52. Rushdi AM. Logic design of NAND (NOR) circuits by the entered-map-factoring method. Microelectronics Reliability. 1987;27(4):693-701.
53. Rushdi AM, Ba-Rukab OM. A map procedure for two-level multiple-output logic minimization. In Proceedings of the Seventeenth National Computer Conference. 2004;521-532.
54. Rushdi AM, Ba-Rukab OM. A purely map procedure for two-level multiple-output logic minimization. International Journal of Computer Mathematics. 2007;84(1):1-10.
55. Rathore TS, Sanila KS. A comparison of two methods for realizing minimal function of several logic variables, SSRG International Journal of Electronics and Communication Engineering. 2021;8(1):6-11.
56. Lee SC. Modern Switching Theory and Digital Design, Prentice-Hall, Englewood Cliffs, New Jersey, NJ, USA; 1978.
57. Halder AK. Karnaugh map extended to six or more variables. Electronics Letters. 1982;18(20):868-870.
58. Motil JM. Views of digital logic & probability via sets, numberings; 2017. Available:http://www.csun.edu/~jmotil/ccSetNums2.pdf.
59. Rushdi AM, Zagzoog SS, Balamesh AS. Derivation of a scalable solution for the problem of factoring an n-bit integer. Journal of Advances in Mathematics and Computer Science. 2019;30(1):1-22.
60. Rushdi AMA, Alsayegh AB. Reliability analysis of a commodity-supply multi-state system using the map method. Journal of Advances in Mathematics and Computer Science. 2019;31(2):1-17.
61. Rushdi AMA, Zagzoog SS. Logical design of n-bit comparators: Pedagogical insight from eight-variable Karnaugh maps. Journal of Advances in Mathematics and Computer Science. 2019;32(3), 1-20.
62. Rushdi AMA, Zagzoog SS. Utilization of Eight-Variable Karnaugh Maps in the Digital Design of n-bit Comparators. Chapter 5 in Theory and Practice of Mathematics and Computer Science Vol. 8, Book Publisher International, Hooghly, West Bengal, India. 2020:68-96.
63. Rushdi AM, Ghaleb FA. Reliability characterization of binary-imaged multi-state coherent threshold systems. International Journal of Mathematical, Engineering and Management Science (IJMEMS). 2020;6(1):309-321.
64. Rushdi AM, Ghaleb FA. Boolean-based symbolic analysis for the reliability of coherent multi-state systems of heterogeneous components. Journal of King Abdulaziz University: Computing and Information Technology Sciences. 2020;9(2):1-25.
65. Lu Z, Liu G, Liao R. A pseudo Karnaugh mapping approach for datasets imbalance. In E3S Web of Conferences (ICERSD 2020), EDP Sciences, 2021;236(04006):1-5.
66. Rushdi AM, Alsayegh AB. Karnaugh-Map Analysis of a Commodity-Supply Multi-State Reliability System. Chapter 3 in Theory and Practice of Mathematics and Computer Science. 2021;8:20-37.
67. Rushdi AM, Alsalami OM. Multistate Reliability Evaluation of Communication Networks via Multi-Valued Karnaugh Maps and Exhaustive Search. Chapter 10 in Advanced Aspects of Engineering Research. 2021;1:114-136.
68. Rushdi AM, Alsalami OM. Reliability analysis of flow networks with an ecological perspective. Network Biology. 2021;11(1):1-28.
69. Rushdi AM, Alsalami OM. Analyzing capacitated networks via Boolean-based coherent pseudo-Boolean functions. 2021;11(2):97-124.

Peer-review history:
The peer review history for this paper can be accessed here:
http://www.sdiarticle4.com/review-history/69065