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The Length-Biased Weighted Erlang Distribution

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Authors' contributions

This work was carried out in collaboration among all authors. Author HMR introduced the idea in a methodically structure, did the data analysis and drafted the manuscript. Author SAO assisted in building the study design and also did the final proofreading. Author AAM managed the analyses of the study. All authors read and approved the final manuscript

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Abstract

We introduce and study a new continuous model so called, the length-biased weighted Erlang (LBWE) distribution. Some mathematical properties of the new distribution such as, raw and incomplete moments, generating functions, Rényi of entropy, order and record statistics are investigated. The moments and maximum likelihood estimates are obtained for the model parameters. An application to real data set is used to illustrate the usefullness of the new model.

Keywords: Erlang distribution; weighted distribution; maximum likelihood estimation; moments estimation; order statistics; record statistics.

1 Introduction

The Erlang distribution has different applications such as queueing theory, mathematical biology and stochastic processes. It was first introduced by Erlang [1] as the distribution of waiting time and message length in telephone traffic. The probability density function (pdf) of the Erlang distribution is given by

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$$f(x;k,\theta)=\frac{x^{k-1}\,e^{-x/\theta}}{\theta^{k}\,\Gamma(k)},\ x>0,\ \ \theta>0,\ \ k=1,2,3,... \tag{1}$$

where θ and k are the scale and shape parameters respectively. Numerous studies have discussed the Erlang distribution such as, Harischandra and Rao [2] pointed out the classical inferences for the Erlangian queue. Bhattacharyya and Singh [3] obtained a Bayes estimator for the Erlangian queue based on two prior distributions. Suri and Bhushan [4] used the Erlang distribution to design a simulator for time estimation of project management process. Haq and Dey [5] derived the Bayes estimates of the parameters of Erlang distribution based on various informative priors. Khan and Jan [6] used different generalized truncated prior distributions to obtain Bayes estimates for the Erlang distribution. Reyad et al. [7] obtained the quasi-Bayes, QE-Bayes, quasi-hierarchical Bayes and quasi-empirical Bayes estimates for the scale parameter of Erlang distribution based on symmetric and different asymmetric loss functions.

The idea of weighted distributions was started by Fisher [8]. Different studies on weighted distributions were published, for example. Gupta and Tripathi [9] introduced the weighted bivariate logarithmic series distribution. Shaban and Boudrissa [10] presented the length biased Weibull distribution. Das and Roy [11] pointed out the length biased weighted generalized Rayleigh distribution. Rashwan [12] discussed the generalized Gamma length biased distribution. Ahmed et al. [13] suggested the length biased weighted Lomax distribution.

Cosider X is a random variable has probability density function (pdf); $f(x)$, then the weighted distribution of the weighted random variable X_w is given from

$$f_w(x)=\frac{w(x)\,f(x)}{E(w(x))},\qquad x>0 \tag{2}$$

where $w(x)$ is a positive weight function and $E(w(x))=\int_0^\infty w(x)\,f(x)\,dx$, $0<E(w(x))<\infty$. We may get various weighted distributions based on different selection of $w(x)$. If $w(x)=\mathrm{x}$, then the resulting distribution is called length-biased distribution with pdf given by

$$f_l(x)=\frac{x\,f(x)}{E(x)},\qquad x>0 \tag{3}$$

where $E(x)=\int_0^\infty x\,f(x)\,dx$, $0<E(x)<\infty$.

This paper suggest and study a new continuous model namely length-biased weighted Erlang (LBWE) distribution Furthermore, the model parameters of the new distribution are estimated using the moments and maximum likelihood methods. A real example is applied to show the flexability of the new model.

The rest of the study is as follows. In Section 2, the length-biased weighted Erlang (LBWE) distribution is defined and some corresponding survival functions. In Section 3, the asympototic behaviour of the new distribution is presented. In Section 4, some mathematical properties are studied. In Section 5, the moment and maximum likelihood estimates for the model parameters are obtained. In Section 6, a real application for the LBWE distribution is applied. Concluding remarks are presented in the last Section.

2 The LBWE Distribution

We define the the length-biased weighted Erlang distribution and some associated reliability functions in this section

The mean of the Erlang distribution given in Eq. (1) is

$$E(x) = \theta k \tag{4}$$

Then, the pdf of length-biased weighted Erlang (LBWE) distribution is derived by using Eqs. (1) and (4) in Eq. (3) to be

$$f(x;k,\theta) = \frac{x^k\, e^{-x/\theta}}{\theta^{k+1}\,\Gamma(k+1)}, \qquad x>0,\ \theta>0,\ k=1,2,3,... \tag{5}$$

The cdf corresponding to Eq. (5) is

$$F(x;k,\theta) = \frac{1}{\theta^{k+1}\,\Gamma(k+1)}\int_0^x y^k\, e^{-y/\theta}\,dy$$

Subsituting $z = y/\theta$, we have

$$F(x;k,\theta) = \frac{1}{\Gamma(k+1)}\int_0^{x/\theta} z^k e^{-z}\,dz = \frac{\gamma(k+1, x/\theta)}{\Gamma(k+1)} \tag{6}$$

where $\gamma(s,x) = \int_0^x t^{a-1}\, e^{-t}\,dt,\ \ a>0,\ b\geq 0$ is the lower incomplete gamma function.

For the survival analysis, the reliability function $R(x)$, hazard function $h(x)$, inverse hazard function $h_r(x)$ and cumulative hazard function $H(x)$ for the LBWE distribution are given respectively as follows:

$$R(x) = 1 - \frac{\gamma(k+1, x/\theta)}{\Gamma(k+1)}, \tag{7}$$

$$h(x) = \frac{x^k\, e^{-x/\theta}}{\Gamma(k+1) - \gamma(k+1, x/\theta)}, \tag{8}$$

$$h_r(x) = \frac{x^k\, e^{\frac{-x}{\theta}}}{\theta^{k+1}\,\gamma(k+1, x/\theta)} \tag{9}$$

And

$$H(x) = -\ln R(x) = -\ln\left\{1 - \frac{\gamma(k+1, x/\theta)}{\Gamma(k+1)}\right\} \tag{10}$$

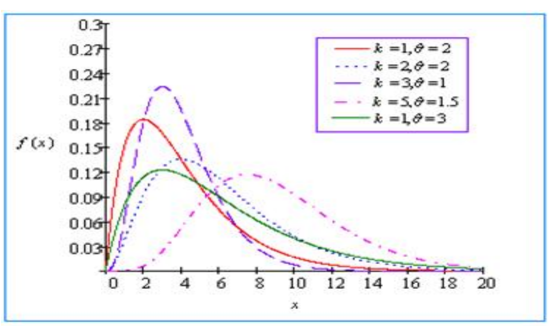

Fig. 1. The pdf of the LBWE distribution for differentvalues of the parameters

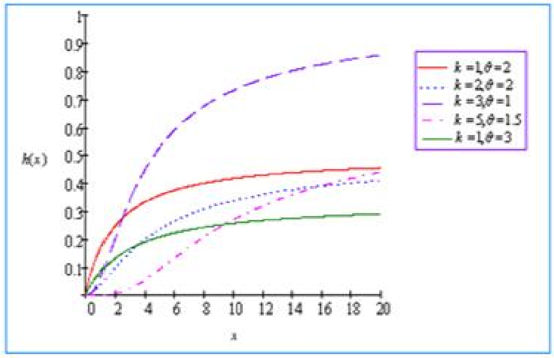

Fig. 2. The h.f. of the LBWE distribution for differentvalues of the parameters

3 Asymptotics

The pdf of the LBWE distribution, $f(x)$ is 0 for both $x \to 0$ and $x \to \infty$, indicating that $f(x)$ is unimodal at $x_0 = \theta k$. Also $f(x)$ is monotonically increasing and decreasing when $x < \theta k$ and $x > \theta k$ respectively. The cdf, $F(x)$ is 0 as $x \to 0$ and 1 as $x \to \infty$. Also, $F(x)$ is a monotonic increasing function of x. The reliability function, $R(x)$ is 1 as $x \to 0$ and 0 as $x \to \infty$ and it is a monotonic decreasing function of x. The hazard function, $h(x)$ is 0 as $x \to 0$ and 1 as $x \to \infty$ and hence it is a monotonic increasing function of x. The shape of the reverse hazard function, $h_r(x)$ appears to be a monotonic decreasing function of x .

4 Mathematical Properties

In this section, we investigate some mathematical properties of the LBWE distribution

4.1 Moments

Suppose X is a random variable distributed according to LBWE distribution, then the rth moments, say μ_r' is given by

$$\mu_r' = E(X^r) = \frac{1}{\theta^{k+1}\,\Gamma(k+1)} \int_0^\infty x^{r+k}\, e^{-x/\theta} dx$$

$$= \frac{\theta^r\,\Gamma(r+k+1)}{\Gamma(k+1)} \tag{11}$$

Substituting $r = 1, 2, 3, 4$ in Eq. (11), we obtain

$$\text{Mean} = \mu_1' = \theta(k+1),$$

$$\mu_2' = \theta^2 (k+1)(k+2),$$

$$\mu_3' = \theta^3 (k+1)(k+2)(k+3)$$

And

$$\mu_4' = \theta^4 (k+1)(k+2)(k+3)(k+4)$$

Also, we have

$$\mu_{r+1}' = E(X^{r+1}) = \frac{1}{\theta^{k+1}\,\Gamma(k+1)} \int_0^\infty x^{r+k+1} e^{-x/\theta} dx$$

$$= \frac{\theta^{r+1}\,\Gamma(r+k+2)}{\Gamma(k+1)} \tag{12}$$

Then, from Eqs. (11) and (12), we can establish a recurrence relation for the non-central moments given by

$$\mu_{r+1}' = \frac{\theta^r}{\Gamma(k+1)} \left[\theta\,\Gamma(r+k+2) - \Gamma(r+k+1)\right] + \mu_r' \tag{13}$$

Therefore, the variance and the standard deviation are given below

$$v(x) = \theta^2 (k+1), \qquad \sigma = \theta\sqrt{k+1} \tag{14}$$

4.2 Incomplete moments

Suppose X is a random variable that follows LBWE distribution, then the rth incomplete moment denoted as $m_r(z)$ can be obtained as follows:

$$m_r(z) = \frac{1}{\theta^{k+1}\,\Gamma(k+1)} \int_0^z x^{r+k} e^{-x/\theta} dx$$

By putting $u = x/\theta$ in above, we get

$$m_r(z) = \frac{\theta^r}{\Gamma(k+1)} \int_0^{z/\theta} u^{r+k}\, e^{-u}\, du$$

Or

$$m_r(z) = \frac{\theta^r \gamma(r+k+1, z/\theta)}{\Gamma(k+1)}, \qquad r = 1,2,3,... \tag{15}$$

4.3 Harmonic mean

The harmonic mean of the LBWE distribution is given by

$$H = \left[E\left(\frac{1}{x}\right)\right]^{-1}$$

$$= \left[\frac{1}{\theta^{k+1}\,\Gamma(k+1)} \int_0^\infty x^{r+k} e^{-x/\theta} dx\right]^{-1} = \theta k \tag{16}$$

4.4 Coefficients of variation, skewness and kurtosis

The coefficients of variation, skewness and kurtosis of the LBWE distribution are given respectively as follows:

$$C.V = \frac{\sigma}{\mu} = \frac{1}{\sqrt{k+1}}, \tag{17}$$

$$\varpi_1 = \frac{\mu_3'}{(\mu_2')^{3/2}} = \frac{k+3}{\sqrt{(k+1)(k+2)}} \tag{18}$$

And

$$\varpi_2 = \frac{\mu_4'}{(\mu_2')^2} = \frac{(k+3)(k+4)}{(k+1)(k+2)} \tag{19}$$

4.5 Generating functions

The moment generating function of the LBWE distribution, say $M_x(t)$ can be obtained as follows:

$$\begin{aligned} M_x(t) = E(e^{tx}) &= \frac{1}{\theta^{k+1}\,\Gamma(k+1)}\int_0^\infty e^{tx} x^k e^{-x/\theta} dx \\ &= \frac{1}{\theta^{k+1}\,\Gamma(k+1)}\int_0^\infty x^k\, e^{-x(1-\theta t)/\theta} dx \end{aligned}$$

By putting $u = x(1-\theta t)/\theta$, we obtain

$$M_x(t) = (1-\theta t)^{-(k+1)} \tag{20}$$

Similarly, the probability generating function of the LBWE, say $M_{[x]}(t)$ distribution can be derived as

$$M_{[x]}(t) = E(t^x) = \frac{1}{\theta^{k+1}\,\Gamma(k+1)}\int_0^\infty t^x x^k\, e^{-x/\theta} dx$$

Using $t^x = \sum_{h=0}^{\infty} \frac{(\ln t)^h x^h}{h!}$, we get

$$M_{[x]}(t) = (1-\theta \ln t)^{-(k+1)} \tag{21}$$

We can obtain the factorial moments $M_{[r]} = E\left[x(x-1)(x-2)...(x-r+1)\right]$ from Eq. (21) according to

$$\left.\frac{\partial M_{[x]}^r(t)}{\partial t^r}\right|_{t=1} = M_{[r]}, r = 1,2,...$$

Consequently, we have

$$M_{[1]} = \theta(k+1)$$

And

$$M_{[2]} = \theta(k+1)\left[\theta(k+2)-1\right],$$

Moreover, the characteristic function of the LBWE distribution is given by

$$\Phi_x(t) = E(e^{itx}) = \frac{1}{\theta^{k+1}\,\Gamma(k+1)}\int_0^{\infty} e^{itx} x^k\, e^{-x/\theta} dx$$
$$= (1-i\theta t)^{-(k+1)} \tag{22}$$

Furthermore, the cumulant generating function of the LBWE denoted as $K_x(t)$ distribution is given by

$$K_x(t) = \ln \Phi_x(t) = -(k+1)\ln(1-i\theta t)$$

Expanding the logarithmic series $\ln(1-i\theta t)$, we get

$$K_x(t) = -(k+1)\left[i\theta t + \frac{(i\theta t)^2}{2} + \frac{(i\theta t)^3}{3} + \frac{(i\theta t)^4}{4} + \ldots\right] \tag{23}$$

We can obtain the central moments; $M_r = E(x-\mu)^r$ from Eq. (23) as the following:

$$M_2 = \text{coefficient of } \frac{(it)^2}{2!} = \theta^2(k+1),$$

$$M_3 = \text{coefficient of } \frac{(it)^3}{3!} = 2\theta^3(k+1)$$

And

$$M_4 = 3M_2^2 + \text{coefficient of } \frac{(it)^3}{3!} = 3\theta^4(k+1)(k+3)$$

4.6 Rényi entropy

The Rényi entropy is defined as

$$I_R(\eta) = \frac{1}{1-\eta}\left[\log I(\eta)\right],$$

where $I(\eta) = \int f^{\eta}(x)\,dx,\ \eta > 0$ and $\eta \neq 0$. Using Eq. (5) yields

$$I(\eta) = \frac{1}{[\theta^{k+1}\,\Gamma(k+1)]^{\eta}}\int_0^{\infty} x^{k\eta}\, e^{-x\eta/\theta} dx$$

$$=\frac{\theta^{1-k}\Gamma(k\eta+1)}{\eta^{k\eta+1}\Gamma^{\eta}(k\eta+1)}$$

Therefore, the Renyi entropy is given from

$$I_R(\eta)=\frac{1}{1-\eta}\log\left[\frac{\theta^{1-k}\Gamma(k\eta+1)}{\eta^{k\eta+1}\Gamma^{\eta}(k+1)}\right]$$
$$=\frac{1}{\eta-1}\left[(1-k)\log\theta+\log\Gamma(k\eta+1)-(k\eta+1)\log\eta-\eta\log\Gamma(k+1)\right] \quad (24)$$

4.7 Order statistics

Order statistics play a vital role in probability and statistics. Let $x_{1,n}\le x_{2:n},...\le x_{n:n}$ be the ordered sample from a continuous population with pdf $f(x)$ and cdf $F(x)$. The pdf of $X_{r:n}$, the rth order statistics is given by

$$f_{X_{r:n}}(x)=\frac{n!}{(r-1)!(n-r)!}f(x)\left[F(x)\right]^{r-1}\left[1-F(x)\right]^{n-r}$$

Then, the pdf the rth order LBWE random variable $X_{r:n}$ can be obtained by using Eqs. (5) and (6) in above equation to be

$$f_{X_{r:n}}(x)=\frac{n!}{(r-1)!(n-r)!}\,\Omega_1\,x^{-(k+1)}\,e^{-x/\theta} \quad (25)$$

Where

$$\Omega_1=\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j\left[\Gamma(k+1)\right]^{-(j+r)}\left[\gamma\left(k+1,x/\theta\right)\right]^{r+j-1}$$

Therefore, the pdf of the first order LBWE random variable $X_{1:n}$ is given by

$$f_{X_{1:n}}(x)=n\,\Omega_2\,x^{-(k+1)}\,e^{-x/\theta} \quad (26)$$

Where

$$\Omega_2=\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^i\left[\Gamma(k+1)\right]^{-(i+1)}\left[\gamma\left(k+1,x/\theta\right)\right]^{i}$$

Also, the pdf the of the last order LBWE random variable $X_{n:n}$ is given by

$$f_{X_{n:n}}(x)=n\,\Omega_3\,x^{-(k+1)}\,e^{-x/\theta} \quad (27)$$

Where

$$\Omega_3=\left[\Gamma(k+1)\right]^{-n}\left[\gamma\left(k+1,x/\theta\right)\right]^{n-1}$$

Moreover, the joint distribution of two order statistics $X_{r:n} \le X_{s:n}$ is given as

$$f_{X_{r:s:n}}(x_1,x_2)=\frac{n!}{(r-1)!(s-r-1)!(n-s)!}\,f(x_1)\,f(x_2)\left[F(x_1)\right]^{r-1}\left[F(x_2)-F(x_1)\right]^{s-r-1}\left[1-F(x_2)\right]^{n-s}$$

Then for the LBWE distribution, we have

$$f_{X_{r:s:n}}(x_1,x_2)=\frac{n!\,\theta^{-2(k+1)}}{(r-1)!(s-r-1)!(n-s)!}\,\Omega_4\,x_1^k\,x_2^k\,\exp\left[(-1/\theta)(x_1+x_2)\right] \tag{28}$$

Where

$$\Omega_4=\sum_{i=0}^{n-s}\sum_{j=0}^{s-r-1}\binom{n-s}{i}\binom{s-r-1}{j}(-1)^{i+j}\left[\Gamma(k+1)\right]^{-(s+i)}\left[\gamma\left(k+1,x_1/\theta\right)\right]^{r+j-1}\left[\gamma\left(k+1,x_2/\theta\right)\right]^{s+i-(j+r)-1}$$

4.8 Record statistics

Record values and the associated statistics are of great importance in many real life applications such as hydrology, industrial stress, athletic events and meteorological analysis. Let $X_1, X_2, ..., X_n$ be a sequence of independent and identically distributed (iid) random variables having cdf $F(x)$ and pdf $f(x)$. Set $Y_n = \max(\min)\{X_1, X_2, ..., X_n\}$ for $n \ge 1$. We say X_j is an upper (lower) record value of this sequence if $Y_j > (<)\, Y_{j-1}, j > 1$. Thus X_j will be called an upper (lower) record value if its value exceeds (is lower than) that of all previous observations.

The pdf of $X_{U(r)}$, the rth upper record is given as (see Ahsanullah [14] and Arnold et al. [15])

$$f_{X_{U(r)}}(x)=\frac{1}{(r-1)!}\left[R(x)\right]^{r-1} f(x) \tag{29}$$

Where

$$R(x)=-\ln\left[1-F(x)\right]$$

Then, the pdf the rth upper record LBWE random variable $X_{U(r)}$ can be obtained by using Eqs. (5) and (6) in Eq. (31) to be

$$f_{X_{U(r)}}(x)=\frac{1}{(r-1)!\,\Gamma(k+1)\,\theta^{k+1}}\,\Omega_5^{r-1}\,x^k\,e^{-x/\theta} \tag{30}$$

Where

$$\Omega_5=\ln\left[\Gamma(k+1)\right]-\ln\left[\Gamma(k+1)-\gamma\left(k+1,x/\theta\right)\right]$$

Also, the joint distribution of the first n upper record values $x \equiv \left(x_{U(1)}, x_{U(2)}, ..., x_{U(n)}\right)$ is given by (see Ahsanullah [14]) as

$$f_{1,2,...,n}(x_{U(1)}, x_{U(2)},..., x_{U(n)}) = f(x_{U(n)}) \prod_{i=1}^{n-1} \frac{f(x_{U(i)})}{1-F(x_{U(i)})}$$

Then, for the LBWE distribution we get

$$f_{1,2,...,n}(x_{U(1)}, x_{U(2)},..., x_{U(n)}) = \frac{\theta^{-n(k+1)}}{\Gamma(k+1)} \exp\left[(-)\left(\Omega_6 - \Omega_7 + \Omega_8\right)\right] \tag{31}$$

Where

$$\Omega_6 = \frac{1}{\theta}\sum_{i=1}^{n} x_i^k, \quad \Omega_7 = k\sum_{i=1}^{n} \ln x_i, \quad \Omega_8 = \sum_{i=1}^{n-1} \ln\left[\Gamma(k+1) - \gamma\left(k+1, x_i/\theta\right)\right]$$

In addition, the pdf of $X_{L(r)}$, the rth lower record is given as (see Ahsanullah [14] and Arnold et al. [15])

$$f_{X_{U(r)}}(x) = \frac{1}{(r-1)!}\left[H(x)\right]^{r-1} f(x)$$

Where

$$H(x) = -\ln\left[F(x)\right]$$

Then, for the LBWE distribution we get

$$f_{X_{L(r)}}(x) = \frac{\theta^{-(k+1)}}{(r-1)!\,\Gamma(k+1)}\,\Omega_9^{r-1}\, x^k\, e^{-x/\theta} \tag{32}$$

Where

$$\Omega_9 = \ln\left[\Gamma(k+1)\right] - \ln\left[\gamma\left(k+1, x_i/\theta\right)\right]$$

Moreover, the joint distribution of the first n lower record values $x \equiv \left(x_{L(1)}, x_{L(2)},..., x_{L(n)}\right)$ is given by (see Ahsanullah [14]) as

$$f_{1,2,...,n}(x_{L(1)}, x_{L(2)},..., x_{L(n)}) = f(x_{L(n)}) \prod_{i=1}^{n-1} \frac{f(x_{L(i)})}{F(x_{L(i)})}$$

So, for the new model we have

$$f_{1,2,...,n}(x_{L(1)}, x_{L(2)},..., x_{L(n)}) = \frac{\theta^{-n(k+1)}}{\Gamma(k+1)} \exp\left[(-)\left(\Omega_6 - \Omega_7 + \Omega_{10}\right)\right] \tag{33}$$

Where

$$\Omega_{10} = \sum_{i=1}^{n-1} \ln\left[\gamma\left(k+1, x_i/\theta\right)\right]$$

5 Methods of Estimation

In this section, we discuss the moment and maximum likelihood estimates for the LBWE distribution.

5.1 Moment estimation

The moment estimates of θ and k denoted as $\hat{\theta}_{MME}$ and $\hat{k}_{MME}$ respectively can be obtained by equating the population and sample moments as follows:

$$\mu_r' = \frac{1}{n}\sum_{i=1}^{n} x_i^r$$

Then, we have:

$$\theta(k+1) = \bar{x}, \qquad \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{34}$$

And

$$\theta^2 (k+1)(k+2) = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \tag{35}$$

From Eq. (34), we get

$$\theta = \frac{\bar{x}}{(k+1)} \tag{36}$$

Using Eq. (36) in Eq. (35), we obtain

$$\hat{k}_{MME} = \frac{\bar{x}^2}{s^2} - 1, \quad s^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2 \tag{37}$$

Substitution Eq. (37) in Eq. (36), we have

$$\hat{\theta}_{MME} = \frac{s}{\bar{x}} \tag{38}$$

5.2 Maximum likelihood estimation

Let $x_1, x_2, ..., x_n$ be a random sample from the LBWE distribution, then the corresponding likelihood function is given by

$$L(x;\theta,k) = \theta^{-n(\alpha-1)}\,\Gamma^{-n}(k+1)\left(\prod_{i=1}^{n} x_i^k\right)\exp\left[(-1/\theta)\sum_{i=1}^{n} x_i\right] \tag{39}$$

Taking the natural logarithm of Eq. (39), we have

$$\ell = \ln L(x;\theta,k) = -n\ln[\Gamma(k+1)] - n(k+1)\ln + k\sum_{i=1}^{n}\ln x_i - \frac{1}{\theta}\sum_{i=1}^{n}x_i\theta \tag{40}$$

Now, differentiating Eq. (40) with respect to θ and k and then equating these derivatives to zero, we obtain the two non-linear equations given below

$$\frac{\partial\ell}{\partial\theta} = \frac{-n(k+1)}{\theta} + \frac{\sum_{i=1}^{n}x_i}{\theta^2} = 0 \tag{41}$$

And

$$\frac{\partial\ell}{\partial k} = -n\Psi(k+1) - n\ln\theta + \sum_{i=1}^{n}\ln x_i = 0 \tag{42}$$

where $\Psi(x)$ is the digamma function. From Eq. (41), we obtain

$$\theta = \frac{\bar{x}}{k+1} \tag{43}$$

Using Eq. (43) in Eq. (42), we get

$$\Psi(k+1) - \ln(k+1) = \frac{1}{n}\sum_{i=1}^{n}\ln x_i - \ln\bar{x} \tag{44}$$

Solving the last equation numerically by using Newton Raphson iteration technique, we obtain the maximum likelihood estimator of k denoted as $\hat{k}_{MLE}$ and then substituting in Eq. (43) to obtain the maximum likelihood estimator of θ denoted as $\hat{\theta}_{MLE}$.

For interval estimation and testing of hypotheses of the model parameters (θ, k), the 2×2 observed information matrix is given below

$$J(\Theta) = \begin{bmatrix} J_{\theta\theta} & J_{\theta k} \\ J_{k\theta} & J_{kk} \end{bmatrix}$$

With

$$J_{\theta\theta} = \frac{\partial^2\ell}{\partial\theta^2} = \frac{n}{\theta^2}\left[k+1-\frac{2\bar{x}}{\theta}\right],$$

$$J_{\theta k} = J_{k\theta} = \frac{\partial^2\ell}{\partial\theta\partial k} = \frac{-n}{\theta}$$

And

$$J_{kk} = \frac{\partial^2\ell}{\partial k^2} = -n\left[\Psi^{(1)}(k+1) - \left(\Psi(k+1)\right)^2\right]$$

where $\Psi^{(n)}(x)$ is the polygamma function.

6 Application

In this section, we provide an application of the LBWE distribution to real data set. The data set represents the relief times of 20 patients receiving an analgesic which are given by Gross and Clark [16]. The data are: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2. These data set is previously studied by Rodrigues et al. [17] and Mead [18].

We use the previous data set to compare the fit of the new model, LBWE distribution with the Erlang distribution.

First, we obtain the maximum likelihood estimates (MLEs) for the parameters of each model and then comparing the results within goodness-of-fit statistics AIC (Akaike information criterion), AICC (corrected Akaike information criterion), CAIC (consistent Akaike information criterion) and BIC (Bayesian information criterion). The better model corresponds to smaller AIC, AICC, CAIC and BIC values.

where

$$AIC = 2p - 2\hat{\ell}(.), \qquad AICC = AIC + \frac{2p(p+1)}{n-p-1}$$

and

$$CAIC = \frac{2pn}{n-p-1} - 2\hat{\ell}(.), \qquad BIC = p\log(n) - 2\hat{\ell}(.)$$

where $\hat{\ell}(.)$ denotes the log-likelihood function evaluated at the MLEs, p is the number of parameters, and n is the sample size. The MLEs and the values of AIC, AICC, CAIC and BIC displayed in Table 1.

Table 1. MLEs for LBWE and Erlang models and the statistics AIC, AICC, CAIC and BIC

Model	Estimates		Statistics				
	$\hat{\theta}$	$\hat{k}$	$-2\hat{\ell}$	AIC	AICC	CAIC	BIC
LBWE	1.238	-2.535	141.68	145.68	146.386	146.386	147.671
Erlang	8608	0.124	146.71	150.71	151.416	151.416	152.701

From Table 1, it has been observed that the LBWE model has the smallest values for the AIC, AICC, CAIC and BIC statistics as compared with the Erlang distribution. Consequently, we can conclude that the LBWE distribution provides a significantly better fit than Erlang distribution.

7 Conclusion

This paper proposses a new continuous model called length-biased weighted Erlang (LBWE) distribution. Basic properties of this distribution such as, the mean, variance, coefficient of variation, harmonic mean, moments, skewness, kurtosis, generating functions, reliability analysis, Rényi of entropy, order statistics and record statistics have been studied. The parameters of this distribution are estimated by using the moments and maximum likelihood techniques. A real application is used to show that the the new model can produce a better fit than the Erlang distribution.

Competing Interests

Authors have declared that no competing interests exist.

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