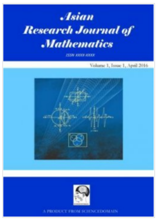

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Maximum Likelihood Estimation in Nonlinear Fractional Stochastic Volatility Model

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

We study the strong consistency and asymptotic normality of the maximum likelihood estimator (MLE) of a drift parameter in a stochastic volatility model when both the asset price process and the stochastic volatility are driven by independent fractional Brownian motions. Long memory in volatility is a stylized fact. We compute the nonlinear filter in the MLE using *Kitagawa algorithm.*

Keywords: *Fractional Brownian motion; stochastic volatility model; maximum likelihood estimate; strong consistency, asymptotic normality; nonlinear filtering; long-range dependence.*

1 Introduction

In mathematical finance, it is well accepted that volatility of a stock price is a stochastic process, not a constant. It is also known that volatility has long memory and clusters on high level (see [1]). One way of modeling long memory is superposition of Ornstein-Uhlenbeck (supOU) processes as volatility models. The class of supOU processes can capture extremal clusters and long range dependence.

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We consider volatility as a continuous model satisfying a stochastic differential equation driven by a persistent fractional Brownian motion. Long memory in volatility a stylized fact in finance due to volatility clustering and persistence. Empirical study shows that volatility has long memory in the sense that the empirical autocorrelation function decreases slower than exponential. Hence parameter estimation in stochastic volatility model with long memory is an important problem in mathematical finance. But the difficulty arises from the fact that volatility is latent as it is not observed in the market. Hence the parameters must be estimated from the corresponding stock price observations. Parameter estimation in directly observed stochastic differential equations is extensively studied in [2] and [3]. Models with fractional Brownian motion and fractional Levy processes (see [4]) as diving terms have attracted recent attention.

The fractional Brownian motion (fBm, in short), which provides a suitable generalization of the Brownian motion, is one of the simplest stochastic processes exhibiting long range-dependence. It was introduced in [5] and later on studied in [6] and [7].

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which all random variables and processes below are defined. A normalized fractional Brownian motion $(W_t^H, t \geq 0)$ with Hurst parameter $H \in (0,1)$ is a centered Gaussian process with continuous sample paths whose covariance kernel is given by

$$E(W_t^H W_s^H) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}), \quad s,t \geq 0.$$

The process is self similar (scale invariant) and it can be represented as a stochastic integral with respect to standard Brownian motion. For $H = \frac{1}{2}$, the process is a standard Brownian motion. For $H \neq \frac{1}{2}$, the fBm is not a semimartingale and not a Markov process, but a Dirichlet process, see for instance, [8]. The increments of the fBm are negatively correlated for $H < \frac{1}{2}$ and positively correlated for for $H > \frac{1}{2}$ and in this case they display long-range dependence. The parameter H which is also called the self similarity parameter, measures the intensity of the long range dependence. The estimation of the parameter H based on observations of the fractional Brownian motion has been studied, see, e.g., [9] and the references therein. However, we assume H to be known.

Hence for $H \neq \frac{1}{2}$, the classical theory of stochastic integration with respect to semimartingales is not applicable to stochastic integration with respect to fBm. Now there exist several approaches to stochastic integration with respect to fBm, see for instance, classical Riemann sum approach ([10], [11], [12], [13]), Malliavin calculus approach ([14], [15], [16], [17], [18], [19]), Wick product approach ([20]), pathwise calculus ([21], [22]), Dirichlet calculus ([8]).

The problem of optimal filtering of a signal when the noise is driven by standard Brownian motion was studied in [23]. Parameter estimation in such partially observed systems was studied in [24], [25], [26], [27], [28] and [29].

The problem of optimal filtering of a signal when the noise is driven by fractional Brownian motion was studied in [30], [12], [31], [32], [16], [33] and [34].

As far as estimation of unknown parameter in fractional system is concerned, maximum likelihood estimator (MLE) of the constant drift parameter of a fractional Brownian motion was obtained in [15] who developed stochastic analysis of the fBm in a Malliavin calculus framework. In [36], Norros, Valkeila and Virtamo studied the properties of the MLE of the constant drift parameter of fBm using martingale tools. They showed that the MLE is unbiased and normally distributed. They also showed that the MLE is strongly consistent and proved a law of the iterated logarithm as $T \to \infty$. The problem was generalized in [31] to a stochastic differential equation driven by fBm with drift and the diffusion coefficient being nonrandom functions and the unknown parameter in the drift coefficient. In [31], Le Breton obtained the best linear unbiased estimator (BLUE) of the drift parameter which coincides with the MLE. He also obtained the least squares estimator (LSE) and compared the relative efficiency of the LSE and the BLUE.

Nonlinear fractional diffusions have vast applications in finance, engineering and biology. Our aim in this paper is to give an algorithm for the approximation of the nonlinear filter, computation of the maximum likelihood estimation and then to study the asymptotic properties of the MLE of a parameter appearing linearly in the drift coefficient of a nonlinear stochastic differential equation driven by fBm when the signal process is a nonlinear diffusion process.

The paper is organized as follows: In Section 2, we prepare model, assumptions and preliminaries. In Section 3, we give the main results of the paper on strong consistency and asymptotic normality of the MLE. In section 4, we give a conclusion of the paper.

2 Model and Assumptions

On the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ satisfying the usual hypotheses, consider the stochastic differential equations

$$\begin{aligned} dY_t &= \theta f\left(t, \sigma_t^2\right) dt + g(\sigma_t^2)\; dW_t^H, \\ d\sigma_t^2 &= a\left(t, \sigma_t^2\right) dt + b\left(t, \sigma_t^2\right) dV_t^H,\; t \in [0, T], \\ Y_0 &= \xi,\;\; \sigma_0^2 = \eta, \end{aligned}$$

where W^H and V^H, $H \in (\frac{1}{2}, 1)$ are independent fractional Brownian motions such that the pair (η, ξ) is independent of (V^H, W^H).

The unknown parameter $\theta \in \Theta$ which is an open subset in $\mathbb{R}$ needs to be estimated on the basis of observation of the asset price process $\{Y_t\}$. The functions f, g, a, b are known satisfying the following properties:

We assume that the functions $f : [0, T] \times \mathbb{R}^+ \to \mathbb{R}$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy (A1):

(A1) (i) For any $R > 0$, there exists $K_R > 0$ such that
$|f(t, x) - f(t, y)| \le K_R|x - y|$ for all $t \in [0, T]$ and for all $|x|, |y| \le R$.
(ii) There exist a function $f_0 \in L_p[0, T]$, $1 \le p < \infty$ and $K > 0$ such that
$|f(t, x)| \le K|x| + f_0(t)$ for all $(t, x) \in [0, T] \times \mathbb{R}$.
(iii) g is differentiable and there exists $K > 0$ such that
$|g(x) - g(y)| \le K|x - y|$ for all $x, y \in \mathbb{R}$.
(iv) For any $R > 0$, there exists $M_R > 0$ such that the derivatives of g are local Hölder continuous in x: there exists $0 < \kappa \le 1$ such that
$|g'(x) - g'(y)| \le M_R|x - y|^\kappa$ for all $t \in [0, T]$, $|x|, |y| \le R$.

We assume that the functions $a : [0, T] \times \mathbb{R}^+ \to \mathbb{R}$ and $b : [0, T] \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfy (A2):

(A2) (i) For any $R > 0$, there exists $K_R > 0$, such that
$|a(t, x) - a(t, y)| \le K_R|x - y|$ for all $t \in [0, T]$ and for all $|x|, |y| \le R$.
(ii) There exists a function $a_0 \in L_p[0, T]$, $1 \le p < \infty$ and $K > 0$ such that
$|a(t, x)| \le K|x| + a_0(t)$ for all $(t, x) \in [0, T] \times \mathbb{R}$.
(iii) b is differentiable in x, there exists $K > 0$ such that
$|b(t, x) - b(t, y)| \le K|x - y|$ for all $t \in [0, T]$.
(iv) For any $R > 0$, there exists $M_R > 0$ such that x derivatives of b are local Hölder continuous in x: there exists $0 < \kappa \le 1$ such that

$|b_x(t,x) - b_x(t,y)| \leq M_R|x-y|^\kappa$ for all $t \in [0,T]$, $|x|, |y| \leq R$.
(v) b is local Hölder continuous in time: there exists $0 < \gamma \leq 1$ and a constant $K > 0$ such that $|b(t,x) - b(s,x)| + |b_x(t,x) - b_x(s,x)| \leq K|t-s|^\gamma$ for all $t, s \in [0,T], x \in \mathbb{R}$.

Under the conditions (A1) and (A2), it is known that there exists a unique solution of the SDEs (see [35]).

3 Maximum Likelihood Estimation

Even if fBM are not martingales, there are simple integral transformations which change the fBm to martingales. We shall use the following result of [33] in the sequel.

Theorem 3.1 *Let h be a continuous function from $[0,T]$ to $\mathbb{R}$. Define for $0 < t \leq T$, the function $k_h^t = (k_h^t(s), 0 < s < t)$ by*

$$k_h^t(s) := -\rho_H^{-1} s^{\frac{1}{2}-H} \frac{d}{ds} \int_s^t d\omega\ \omega^{2H-1}(\omega - s)^{\frac{1}{2}-H} \frac{d}{d\omega} \int_0^\omega dz\ z^{\frac{1}{2}-H}(\omega - z)^{\frac{1}{2}-H} h(z)$$

where $\rho_H = \Gamma^2(3/2 - H)\Gamma(2H+1)\sin \pi H$. Then k_h^t satisfies

$$H(2H-1)\int_0^t k_h(s)|s-r|^{2H-2} ds = h(r); \quad 0 < r < t.$$

Define for $0 \leq t \leq T$, $N_t^h := \int_0^t k_h^t(s) dW_s^H$, $\langle N^h \rangle_t := \int_0^t h(s) k_h^t(s) ds$. Then the process $\{N_t^h, 0 \leq t \leq T\}$ is a Gaussian martingale with variance function $\{\langle N^h\rangle_t, 0 \leq t \leq T\}$. For $h = 1$, the function k_h^t is $k_*^t(s) := \tau_H^{-1}(s(t-s))^{\frac{1}{2}-H}$ where $\tau_H := 2H\Gamma(3/2 - H)\Gamma(H + \frac{1}{2})$. Then the corresponding Gaussian martingale is $N_t^* = \int_0^t k_*^t(s) dW_s^H$ with variance function $\langle N^* \rangle_t = \int_0^t k_*^t(s) ds = \lambda_H^{-1} t^{2-2H}$ where $\lambda_H := \frac{2H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(3/2-H)}$.

This theorem was shown in [31]. They have shown that N^* is a Gaussian martingale with variance function $\langle N^* \rangle$. The process N is also a martingale.

The process $N_t^h := \int_0^t k_h^t(s) dW_s^H$ can be understood in the following way.

Let

$$M_t^h := \int_0^t k_h^t(s) dV_s^H.$$

The process $M = (M_t^h, t \geq 0)$ is a Gaussian martingale (see [36]) the *fundamental martingale* whose variance function is $\langle M^h \rangle_t$. Moreover, the natural filtration of the martingale M coincides with the natural filtration of the fBM V_H. Similarly $N = (N_t, t \geq 0)$ stands for the fundamental martingale of W_H.

Consider the canonical space of the process (σ^2, Y). Let $\Omega = \mathcal{C}([0,T]; \mathbb{R}^2)$ be the space of continuous functions from $[0,T]$ into $\mathbb{R}^2$. Consider also the canonical process $(\sigma^2, W^*) = ((\sigma_t^2, W_t^*), t \in [0,T])$ on Ω where $(\sigma_t^2, W_t^*)(x,y) = (x_t, y_t)$ for any $(x,y) \in \Omega$.

The probability measure $\widetilde{\mathbb{P}}$ denotes the unique probability measure on Ω such that defining the variable ξ by $\xi = W_0^*$ and $\widetilde{W} = (\widetilde{W}_t), t \in [0,T]$ by $\widetilde{W}_t = W_t^* - W_0^*, t \in [0,T]$, the pair (σ^2, ξ) is independent of $\widetilde{W}$ and the process $\widetilde{W}$ is a fBm with Hurst parameter H.

The canonical filtration on Ω is $(\mathcal{F}_t, t \in [0,T])$ where $\mathcal{F}_t = \sigma(\{(\sigma_s^2, W_s^*), 0 \leq s \leq t\}) \bigvee \mathcal{N}$ where $\mathcal{N}$ denotes the set of null sets of $(\Omega, \widetilde{\mathbb{P}})$.

Define for all continuous functions $x = (x_t, t \in [0,T])$ the function $h(\theta, x)$ on $[0,T]$ by

$$h(\theta, x)(t) := \frac{\theta f(t,x)}{g(x)}, \quad t \in [0,T].$$

Consider, for any $t \in [0,T]$, the function $k^t_{h(\theta,x)} = (k^t_{h(\theta,x)}(s), 0 < s < t)$ defined from Theorem 3.1 with $h(\theta, x)$ in place of h.

Define the processes $N = (N_t, t \in [0,T])$ and $\langle N \rangle = (\langle N \rangle_t, t \in [0,T])$ from Theorem 3.1 by plugging in the process $h(\theta, x)$ in place of h, i.e., $N_t := N_t^{h(\theta,\sigma^2)}, \ \ \langle N \rangle_t := \langle N^{h(\theta,\sigma^2)} \rangle_t$.

Notice that N_t and $\langle N \rangle_t$ depend only on the values of $\sigma^{2(t)} := (\sigma_s^2, 0 \le s \le t)$.

Define the $(\mathcal{F}_t)$-adapted processes $\langle N, N^* \rangle = (\langle N, N^* \rangle_t, \ t \in [0,T])$ and

$$q(\theta, \sigma^2) = (q_t(\theta, \sigma^2), \ t \in [0,T])$$

by

$$\langle N, N^* \rangle_t := \langle N^{h(\theta,\sigma^2)}, N^* \rangle_t = \int_0^t k^t_*(s) h(\theta, \sigma^2)(s) ds, \quad t \in [0,T])$$

and

$$q_t(\theta, \sigma^2) := q_t^{h(\theta,\sigma^2)} = \frac{d\langle N, N^* \rangle_t}{d\langle N^* \rangle_t}, t \in [0,T].$$

Let $\widetilde{q}_t(\sigma^2) := \frac{q_t(\theta,\sigma^2)}{\theta}$. For $0 \le t \le T$, define the processes

$$\widetilde{N}_t(\theta, x) := \int_0^t k^t_{h(\theta,x)}(s) d\widetilde{W}_s^H, \quad \langle \widetilde{N} \rangle_t(\theta, x) := \int_0^t h(\theta, x)(s) k_h^t(s) ds.$$

Define

$$\Lambda_t(\theta, x) := \exp\left\{ \widetilde{N}_t(\theta, x) - \frac{1}{2} \langle \widetilde{N} \rangle_t(\theta, x) \right\}, t \in [0,T].$$

Define for any $t \in [0,T]$, $\Lambda_t(\theta) := \Lambda_t(\theta, \sigma^2)$. Let $\mathbb{P} := \Lambda_T(\theta)\widetilde{\mathbb{P}}$.

The stochastic integral

$$\widetilde{N}_t(\theta, \sigma^2) = \int_0^t k^t_{h(\theta,\sigma^2)}(s) d\widetilde{W}_s^H$$

exits since σ^2 is a fractional diffusion process and W^H and V^H, $H \in (\frac{1}{2}, 1)$ are independent fractional Brownian motions (see [35]).

Since the stochastic integral

$$\widetilde{N}_t(\theta, \sigma^2) = \int_0^t k^t_{h(\theta,\sigma^2)}(s) d\widetilde{W}_s^H$$

exits, hence $\int_0^t k^t_{h(\theta,\sigma^2)}(s) g^{-1}(\sigma_s^2) dY_s$ exists.

Let $\mathcal{Y}_t := \sigma(\{Y_s, 0 \le s \le t\}), t \in [0,T]$. Define the optimal filter

$$\pi_t(\phi) := \mathbb{E}[\phi(\sigma_t^2)|\mathcal{Y}_t], \quad t \in [0,T]$$

and the unnormalized filter

$$\sigma_t(\phi) := \widetilde{\mathbb{E}}[\phi(\sigma_t^2)\Lambda_t|\mathcal{Y}_t], \quad t \in [0,T].$$

Then the *Kallianpur-Striebel formula* holds: for all $t \in [0,T]$, $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ almost surely

$$\pi_t(\phi) = \frac{\sigma_t(\phi)}{\sigma_t(1)}.$$

Recall that Y is the observation process satisfying

$$dY_t = \theta f\left(t, \sigma_t^2\right) dt + g(\sigma_t^2)\ dW_t^H.$$

Following [34], let us introduce the *fundamental semimartingales* associated with Y, namely Z and Z^* defined by

$$Z_t := \int_0^t k^t_{h(\theta,\sigma^2)}(s) g^{-1}(\sigma_s^2) dY_s, \quad t \in [0,T]$$

and

$$Z_t^* := \int_0^t k_*^t(s) g^{-1}(\sigma_s^2) dY_s, \quad t \in [0,T].$$

where Y is the observation process.

Thus Y can be represented as $Y_t = \int_0^t K_H(t,s) dZ_s$ where

$$K_H(t,s) = H(2H-1)\int_s^t r^{H-\frac{1}{2}}(r-s)^{H-\frac{3}{2}} dr$$

for $0 \le s \le t$ and therefore the natural filtrations of Y and Z coincide. Following [34], the following representation holds:

$$dZ_t = \lambda_H l(t)^* \zeta_t d\langle N\rangle_t + dN_t, \quad Z_0 = 0,$$

where $\zeta = (\zeta_t, t \ge 0)$ is the solution of the stochastic differential equation

$$d\zeta_t = \theta \lambda_H B(t) \zeta_t d\langle M\rangle_t + r(t) dM_t, \quad \zeta_0 = 0,$$

with

$$l(t) = \begin{pmatrix} t^{2H-1} \\ 1 \end{pmatrix}, \ B(t) = \begin{pmatrix} t^{2H-1} & 1 \\ t^{4H-2} & t^{2H-1} \end{pmatrix}, \ r(t) = \begin{pmatrix} 1 \\ t^{2H-1} \end{pmatrix}.$$

The processes Z and Z^* are semimartingales with the following decomposition:

$$Z_t^* = \int_0^t q_s(\theta, \sigma^2) d\langle N^*\rangle_s + N_t^*, \quad t \in [0,T],$$

and

$$Z_t = \int_0^t q_s^2(\theta, \sigma^2) d\langle N^*\rangle_s + \int_0^t q_s(\theta, \sigma^2) dN_s^*, \quad t \in [0,T],$$

Hence we get the integral representation of Z in terms of Z^*:

$$Z_t = \int_0^t q_s(\theta, \sigma^2) dZ_s^*, \quad t \in [0,T].$$

The natural filtration of Z and σ^2 coincide. Introduce the process $\nu = (\nu_t, t \in [0,T])$ defined by

$$\nu_t := Z_t^* - \int_0^t \pi_s(q) d\langle N^*\rangle_s, \quad t \in [0,T]$$

which plays the role of *innovation process* in the usual situation where the noises are Brownian motions.

The conditional expectation $\pi_t(\zeta) = E_\theta(\zeta_t|\mathcal{F}_t)$ satisfies the equation

$$d\pi_t(\zeta) = (\theta\lambda_H B - \lambda_H^2 \gamma_{\zeta\zeta} ll^*)\pi_t(\zeta)d\langle N\rangle_t + \lambda_H \gamma_{\zeta\zeta} l dZ_t, \quad \pi_0(\zeta) = 0.$$

Here

$$\gamma_{\zeta\zeta} = E_\theta(\zeta_t - \pi_t(\zeta))^*(\zeta_t - \pi_t(\zeta))$$

is the covariance of the filtering error which is the unique solution of the Ricatti equation

$$d\gamma_{\zeta\zeta} = (\theta\lambda_H(B\gamma_{\zeta\zeta} + \gamma_{\zeta\zeta}B^*) + rr^* - \lambda_H^2 \gamma_{\zeta\zeta} ll^* \gamma_{\zeta\zeta})\gamma_{\zeta\zeta} d\langle N\rangle_t, \quad \gamma_{\zeta\zeta} = 0.$$

The above equation on $\pi_t(\zeta)$ can be written as

$$d\pi_t(\zeta) = \theta\lambda_H B\pi_t(\zeta)d\langle N\rangle_t + \lambda_H \gamma_{\zeta\zeta} l d\nu_t$$

where the innovation process ν_t is defined by

$$d\nu_t = dZ_t - \lambda_H l(t)^* \pi_t(\zeta)d\langle N\rangle_t, \ \nu_0 = 0.$$

Recall the notation $\pi_s(q) := \mathbb{E}[q_s(\theta, \sigma^2)|\mathcal{Y}_s], s \in [0, t]$. The particular case of unnormalized filter is

$$\widetilde{\Lambda}_t(\theta) := \sigma_t(1) = \widetilde{\mathbb{E}}[\Lambda_t|\mathcal{Y}_t], \ t \in [0, T].$$

By Proposition 3 in [33], we have

$$\widetilde{\Lambda}_T(\theta, \mathcal{Y}_t) = \exp\left\{\theta \int_0^T \pi_s(\widetilde{q})dZ_s^* - \frac{\theta^2}{2}\int_0^T \pi_s^2(\widetilde{q})d\langle N^*\rangle_s\right\}.$$

Thus the maximum likelihood estimator (MLE) of θ is given by

$$\hat{\theta}_T \quad := \quad \arg\max_{\theta\in\Theta} \widetilde{\Lambda}_T(\theta) = \frac{\int_0^T \pi_s(\widetilde{q})dZ_s^*}{\int_0^T \pi_s^2(\widetilde{q})d\langle N^*\rangle_s}.$$

In a linear state-space system, it is well known that Kalman filter is an algorithm for the exact computation of the conditional p.d.f. of the signal given the observation (and the initial conditions). Except for some very specific cases, such as the fractional Cox-Ingersoll-Ross (fCIR) models which has noncetral chi-square transition density, exact computation of the likelihood function is not possible in nonlinear models. Since we have a nonlinear state-space type model, we seek numerical or simulated methods. One such computational algorithm is Kitagawa's algorithm to compute the conditional p.d.f. recursively.

In [37], Kitagawa suggested a *linear spline technique* for approximating the nonlinear filter. The basic idea is to use the relevant probability density functions for each period by piecewise linear functions. The accuracy of density approximation depends on the number of nodes used for the piecewise linear approximation and the number of nodes is limited by the computational demands. We use the Kitagawa algorithm to compute the MLE. We approximate the filter

$$\pi_s(\widetilde{q}) = \mathbb{E}[\widetilde{q}_s(\theta, \sigma_s^2)|\mathcal{Y}_s]$$

using Kitagawa algorithm given below.

The *prediction-update algorithm* proceeds as follows:

Prediction step:

Recall that for $s \in [0,T]$, $\mathcal{Y}_s := \sigma(\{Y_v, 0 \leq v \leq s\})$. *The prediction step determines the conditional probability density function of the state given the observation.*

Let $\widetilde{p}(\sigma_s^2|\sigma_u^2)$ *be the conditional density of* σ_s^2 *given* σ_u^2 *where* $u < s$. *Then*

$$\mathbb{E}[\widetilde{q}_s(\theta,\sigma_s^2)|\mathcal{Y}_u] = \int_{-\infty}^{\infty} \widetilde{p}(\sigma_s^2|\sigma_u^2)\mathbb{E}[\widetilde{q}_s(\theta,\sigma_u^2)|\mathcal{Y}_u]d\sigma_u^2$$

Update step:

If observation becomes available at the forecast time, the update step combines this additional information with the estimate from the prediction step.

Suppose that a new observation Y_s *becomes available. This additional information can be used to produce an updated estimate of the predicted state. This new estimate is summarized by the conditional density function* $\widetilde{p}(\sigma_s^2|\mathcal{Y}_s)$.

Let $r(Y_s|\sigma_s^2)$ *be the conditional density of* Y_s *given* σ_s^2.

By Bayes theorem, for $u < s$,

$$\mathbb{E}[\widetilde{q}_s(\theta,\sigma_s^2)|\mathcal{Y}_s] = \frac{r(Y_s|\sigma_s^2)\mathbb{E}[\widetilde{q}_s(\theta,\sigma_s^2)|\mathcal{Y}_u]}{\int r(Y_s|\sigma_s^2)\mathbb{E}[\widetilde{q}_s(\theta,\sigma_s^2)|\mathcal{Y}_u]d\sigma_s^2}.$$

The update and the prediction step yield the general filtering solution.

Next we prove the strong consistency, law of the iterated logarithm and asymptotic normality of the MLE with a random normalization.

Theorem 3.2 *Under the conditions (A1)-(A2),* $\hat{\theta}_T \to \theta$ *a.s. as* $T \to \infty$, *i.e.,* $\hat{\theta}_T$ *is a strongly consistent estimator of* θ. *Moreover,*

$$\limsup_{T\to\infty} \frac{A_T^{1/2}|\hat{\theta}_T - \theta|}{(2\log\log A_T)^{1/2}} = 1 \ a.s.$$

where $A_T := \int_0^T \pi_s^2(\widetilde{q})d\langle N^*\rangle_s$.

Proof. Note that

$$dZ_t^* = \pi_t(q)d\langle N^*\rangle_t + d\nu_t.$$

From Lemma 3 in [33], we have ν is a continuous Gaussian $(\mathcal{Y}_t, \mathbb{P})$ martingale such that $\langle\nu\rangle = \langle N^*\rangle$.

Hence

$$\hat{\theta}_T = \frac{\int_0^T \pi_s(\widetilde{q})dZ_s^*}{\int_0^T \pi_s^2(\widetilde{q})d\langle N^*\rangle_s} = \theta + \frac{\int_0^T \pi_s(\widetilde{q})d\nu_s}{\int_0^T \pi_s^2(\widetilde{q})d\langle N^*\rangle_s}.$$

Now by the strong law of large numbers for continuous martingales (see [38] or Theorem 2.6.10 in [39]), the second term in r.h.s. converges to zero a.s. as $T \to \infty$. Hence strong consistency follows.

Since the deviation $\hat{\theta}_T - \theta$ is obtained from a stochastic integral with respect to a continuous martingale, the law of the iterated logarithm follows from Corollary 1.1.12 in [38]. □

Theorem 3.3 *Under the conditions (A1)-(A2),*

$$A_T^{1/2}(\hat{\theta}_T - \theta) \quad \xrightarrow{\mathcal{D}} \quad \mathcal{N}(0,1) \quad \text{as } T \to \infty.$$

Proof. Note that

$$A_T^{1/2}(\hat{\theta}_T - \theta) = \frac{\int_0^T \pi_s(\widetilde{q}) d\nu_s}{\sqrt{\int_0^T \pi_s^2(\widetilde{q}) d\langle N^* \rangle_s}}.$$

By the central limit theorem for stochastic integrals with respect to Gaussian martingales (see [39]), the r.h.s. above converges in distribution to $\mathcal{N}(0,1)$ as $T \to \infty$. □

4 Conclusion

Fractional stochastic volatility model proposed in the paper is important in practice as it volatility time varying and random and it captures the long memory property. The challenge lies in the fact that volatility is not observed in the market. In order to do option price for such models the parameters of the volatility model must be estimated from the asset price data. We gave an algorithm for the drift estimator of the volatility model using computational maximum likelihood approach using Kitagawa algorithm and nonlinear filtering theory. We obtained strong consistency and asymptotic normality of the maximum likelihood estimator when the process is observed in a large time interval.

Competing Interests

Author has declared that no competing interests exist.

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