British Journal of Mathematics & Computer Science

22(5): 1-12, 2017; Article no.BJMCS.34009

ISSN: 2231-0851

Geodesically Complete Lie Algebroid

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/BJMCS/2017/34009
Editor(s):
(1) Jacek Dziok, Institute of Mathematics, University of Rzeszow, Poland.
Reviewers:
(1) Francisco Bulnes, Tecnolgico de Estudios Superiores de Chalco, Mexico.
(2) Ali Mutlu, Manisa Celal Bayar University, Turkey.
Complete Peer review History: http://www.sciencedomain.org/review-history/19555

***Received:** 9th May 2017*
***Accepted:** 29th May 2017*
***Published:** 15th June 2017*

Original Research Article

Abstract

In this paper we introduce the notion of geodesically complete Lie algebroid. We give a Riemannian distance on the connected base manifold of a Riemannian Lie algebroid. We also prove that the distance is equivalent to natural one if the base manifold was endowed with Riemannian metric. We obtain Hopf Rinow type theorem in the case of transitive Riemannian Lie algebroid, and give a characterization of the connected base manifold of a geodesically complete Lie algebroid.

Keywords: Lie algebroid; Riemannian metric and distance; geodesically complete structure.

2010 Mathematics Subject Classification: 53B20, 53C12, 53C22, 53D17, 58D17.

1 Introduction

Lie groupoids and Lie algebroids are an important and active domain of research in differential geometry [1, 2, 3, 4, 5].

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Lie algebroids were first introduced by J. Pradines as the infinitesimal counterpart of the notion of Lie groupoid. The notion generalizes the tangent bundle and Lie algebra. Hence, one can study on Lie algebroids a lot of notion of differential geometry. As examples, we have covariant derivative by Fernandes [6], Lagrangian mechanic by A. Weinstein, integrability by M. Crainic and R.L. Fernandes [7].

M. Boucetta [1] introduced in 2011, the notion of Riemannian metric on Lie algebroid as a generalization of a Riemannian metric on a vector bundle. Hence, he studies the Levi-Civita connection of a Riemannian Lie algebroid and shows the existence of two tensors similars to those introduce by O'Neill in the contexte of Riemannian submersion (see [8] for more detail). He also studies the geodesic flow of Riemannian Lie algebroid. As in the classic case, he defines the Sasaki metric and computes the divergence of geodesic flow with respect to this metric. He also states the first and the second variations formulas and introduces Jacobi sections along a geodesic. He studies the curvature of Riemannian Lie algebroid and generalizes some classics results, namely Mayers theorem. At last, he states the study of integrability of Riemannian Lie algebroids; for instance, he shows that the vanishing of one of the O'Niell's tensors implies the integrability, and he gives a large class of Riemannian Lie algebroids which satisfy this condition.

Our aim in this paper is to rewrite some notions known on Riemannian geometry. Here, we give the notion of geodesically complete Lie algebroid. We will also give a new Riemannian distance on the connected base manifold of a Riemannian Lie algebroid, like in the case of Riemannian geometry. This distance is induced by the Riemannian metric of Lie algebroid. Thus, we give the like Hopf Rinow theorem.

The paper is organised as follow. After an introduction given in the first section, the second section deals with the basic facts on Lie algebroid and Riemannian Lie algebroid. In the third section, we give a caracterisation of A-geodesics curves and their relationship with base manifold's one. The fourth section deals with geodesically complete Lie algebroid. Thus, after introducing the notion of maximal A-geodesics and the notion of geodesically complete Lie algebroid, we give a caracterisation of this class of Lie algebroid. In the last section, we show the existence of Riemannian distance on the connected base of a Riemannian Lie algebroid. This distance is induced by the Riemannian metric of the Lie algebroid. Thus for a transitive Riemannian Lie algebroid $p : A \to M$ with anchor map $\sharp$ and Riemannian metric g, we show that this metric and the classical one obtained by Riemannian manifold are equivalent. Then at last, we give Hopf Rinow type theorem on transitive Riemannian Lie algebroid and its application for caracterising the leaves of a caracteristic foliation.

2 Some Basic Facts on Riemannian Lie Algebroid

Most of notions introduced in this section come from Boucetta [1] and from J.-P. Dufour and N. T. Zung' s book [9].

2.1 Definition and first properties

A Lie algebroid is a vector bundle $p : A \to M$ such that :

- the sections space $\Gamma(A)$ carry a Lie structure $[,]$;
- there is a bundle map $\sharp : A \to TM$ named anchor;
- For all $a, b \in \Gamma(A)$ and $f \in C^\infty(M)$, then

$$[a, fb] = f[a, b] + \sharp(a)(f)b \tag{1}$$

Note that a Lie algebroid is said to be transitif if the anchor is surjective.

The anchor $\sharp$ satisfy:

$$\sharp[a,b] = [\sharp(a), \sharp(b)]$$

where $a, b \in \Gamma(A)$ and the bracket in the right is the natural Lie bracket of vector bundle. We have also:

$$[fa, b] \quad = \quad f[a,b] - \sharp(b)(f)a \tag{2}$$

and

$$[fa, gb] \quad = \quad fg[a,b] + f(\sharp(a)(g))b - g(\sharp(b)(f))a \tag{3}$$

for any $a, b \in \Gamma(A)$ and $f, g \in C^{\infty}(M)$

In [6], R. Fernandes gives a local splitting of a lie algebroid.

Theorem 2.1. *([6])(local splitting) Let $x_0 \in M$ be a point where $\sharp_{x_0}$ has rank q. There exists a system of cordinates $(x_1, \cdots, x_q, y_1, \cdots, y_{n-q})$ valid in a neighborhood U of x_0 and a basis of sections $\{a_1, \cdots, a_r\}$ of A over U, such that*

$$\begin{aligned} \sharp(a_i) &= \partial_{x_i} \quad (i = 1, \cdots, q), \\ \sharp(a_i) &= \sum_j b^{ij} \partial_{y_j} \quad (i = q+1, \cdots, r), \end{aligned}$$

where $b^{ij} \in C^{\infty}(U)$ are smooth functions depending only on the $y's$ and vanishing at x_0 : $b^{ij} = b^{ij}(y^s), b^{ij}(x_0) = 0$. Moreover, for any $i, j = 1, \cdots, r$,

$$[a_i, a_j] = \sum_u C^u_{ij} a_u$$

where $C^u_{ij} \in C^{\infty}(U)$ vanish if $u \leq q$ and satisfy $\sum_{u>q} \frac{\partial C^u_{ij}}{\partial x_s} b^{ut} = 0$.

2.2 A-connection on Lie algebroid

The notion of connection on Lie algebroids were first introduced in the context of Poisson geometry namely by Vaisman in [10] and R. Fernandes in [11, 6]. It's appeared as a natural extension of the usual connection on fiber bundle (covariant derivative).

Let $E \to M$ be a vector bundle. An A-connection on the vector bundle $E \to M$ is an operator $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$ satisfying:

1. $\nabla_{a+b} s = \nabla_a s + \nabla_b s$ for any $a, b \in \Gamma(A)$ and $s \in \Gamma(E)$;
2. $\nabla_a(s_1 + s_2) = \nabla_a s_1 + \nabla_a s_2$ for any $a \in \Gamma(A)$ and $s_1, s_2 \in \Gamma(E)$;
3. $\nabla_{fa} s = f \nabla_a s$ for any $a \in \Gamma(A)$, $\quad s \in \Gamma(E)$ and $f \in C^{\infty}(M)$;
4. $\nabla_a(fs) = f\nabla_a s + \sharp(a)(f)s$ for any $a \in \Gamma(A)$, $\quad s \in \Gamma(E)$ and $f \in C^{\infty}(M)$.

Remark 2.1. *The notion of A-connection is a generalization of the notion of the usual linear connection on a vector bundle. Lot of classic notions associate with covariant derivative can be written in the case of Lie algebroid.*

Definition 2.1. *Let $p : A \to M$ be a Lie algebroid with anchor map $\sharp$. An A-path on A is a smooth path $\alpha : [t_0, t_1] \to A$ such that:*

$$\sharp(\alpha) = \frac{d}{dt} p(\alpha(t)) \tag{4}$$

The curve $\gamma : [t_0, t_1] \to M$ defined by $\gamma(t) = p(\alpha(t))$ is the base path of α.
An A-path α is said to be vertical if $\sharp(\alpha) = 0$ for all $t \in [t_0, t_1]$.

Note that any A-path lies on a fixed leaf of the algebroid.

Hence one can define a space of smooths applications $s : [t_0, t_1] \to E$, which have the same base path with α. This applications are called α-sections and the space of α-sections is denoted $\Gamma(E)_\alpha$. This notion plays a crucial role in the study of parallel transport on Lie algebroid.

Proposition 2.1. *[1] There exists an unique map $\nabla^\alpha : \Gamma(E)_\alpha \to \Gamma(E)_\alpha$ satisfying :*

1. $\nabla^\alpha(c_1 s_1 + c_2 s_2) = c_1 \nabla^\alpha s_1 + c_2 \nabla^\alpha s_2, \quad c_1, c_2 \in \mathbb{R}$,
2. $\nabla^\alpha f s = f' s + f \nabla^\alpha s$ *where* $f : [t_0, t_1] \to \mathbb{R}$ *is a smooth function.*
3. *if* $\tilde{s}$ *is a local section of* E *which extends* s *and* $\sharp(\alpha(t)) \neq 0$ *then* $\nabla^\alpha s(t) = \nabla_{\alpha(t)} \tilde{s} + \frac{d}{dt} s(t)$;
4. *if* $\tilde{s}$ *is a local section of* E *which extend* s *and* α *is vertical then* $\nabla^\alpha s(t) = \nabla_{\alpha(t)} \tilde{s}$.

For introducing the notion of parallel transport, Boucetta sets the folowing definition

Definition 2.2.

- *An α-section s is said to be parallel along α if $\nabla^\alpha s = 0$.*
- *The parallel transport along α is denoted by:*

$$\tau_\alpha^t : E_{\gamma(t_0} \to E_{\gamma(t)},$$

and $\tau_\alpha^t(s_0) = s(t)$ where s is the unique parallel α-section satifying $s(0) = s_0$.

If $\alpha_0 \in A_x$ and s is a section of E in a neighborhood of x one can check easily that

$$\nabla_{\alpha_0} s = \frac{d}{dt}_{|t=0} (\tau_\alpha)^{-1}(s(\gamma(t))) \tag{5}$$

where α is any A-path satisfying $\alpha(0) = \alpha_0$.

Then we can introduced the notion of linear A-connection.

Definition 2.3. *Let $p : A \to M$ be a Lie algebroid. A linear A-connection D is an A-connection on the vector bundle $A \to M$*

If $(x_1, \cdots, x_n)$ is a system of local coordinates in a neighborhood $U \subset M$ in which $\{a_1, \cdots, a_r\}$ is a base of sections of $\Gamma(A)$, then the Christoffel's symbols of the linear connection D can be defined by:

$$D_{a_i} a_j = \sum_{k=1}^{r} \Gamma_{ij}^k a_k.$$

The most interesting fact of this notion is one can ask about her relationship with the natural covariant derivative. The answer giving by Fernandes in [[6]] is relative to the notion of compatibility with Lie algebroid structure.

Definition 2.4. *A linear A-connection D is compatible with the Lie algebroid structure of A if there is a linear connection on TM (covariant derivative) ∇ such that*

$$\sharp D = \nabla \sharp$$

Proposition 2.2. *[7] Every Lie algebroid admits a compatible linear connection.*

Remark 2.2. *There is another notion of compatibility between linear A-connection and Lie algebroid structure introduced by Boucetta in [1] which is less stronger than the above one. A linear A-connection D is strongly compatible with the Lie algebroid structure if, for any A-path α, the parallel transport τ_α preserves $Ker\sharp$. A linear A-connection D is weakly compatible with the Lie algebroid structure if, for any vertical A-path α, the parallel transport τ_α preserves $Ker\sharp$.*

Proposition 2.3. *[1]*

1. *A linear A-connection is strongly compatible with Lie algebroid stucture if and only if, for any leaf* L *and any sections* $\alpha \in \Gamma(A_L)$ *and* $\beta \in \Gamma(Ker\sharp_L)$, $D_\alpha\beta \in \Gamma(Ker\sharp_L)$.
2. *A linear A-connection* D *is weakly compatible with the Lie algebroid structure if and only if, for any leaf* L *and any sections* $\alpha \in \Gamma(Ker\sharp_L)$ *and* $\beta \in \Gamma(Ker\sharp_L)$, $D_\alpha\beta \in \Gamma(Ker\sharp_L)$.

2.3 Riemannian metric

A Riemannian metric on a Lie algebroid $p : A \to M$ is the data, for any $x \in M$, of a scalar product $<,>_x$ on the fiber A_x such that, for any local sections $a, b \in \Gamma(A)$, the function $< a, b >$ is smooth.

Morever, one can define the Levi-civita A-connection which is the linear A-connection D characterized by the following properties:

1. D is metric, i.e., $\sharp(a) < b, c >=< D_a b, c > + < b, D_a c >$,
2. D is torsion free, i.e., $D_a b - D_b a = [a, b]$.

This Levi-civita A-connection satisfay the following formula :

$$\begin{aligned} 2 < D_a b, c > &= \sharp(a) < b, c > +\sharp(b) < a, c > -\sharp(c) < a, b > \\ &+ < [c, a], b > + < [c, b], a > + < [a, b], c > \end{aligned}$$

The Christoffel's symbols of the Levi-civita A-connection are defined, in a local coordinates system $(x_1, \cdots, x_n)$ over a trivializing neighborhood U of M where $\Gamma(A)$ admits a local basis of sections $\{a_1, \cdots, a_r\}$, by:

$$\begin{aligned} \Gamma^k_{ij} &= \frac{1}{2}\sum_{l=1}^{r}\sum_{u=1}^{n} g^{kl}(b^{iu}\partial x_u(g_{jl}) + b^{ju}\partial x_u(g_{il}) - b^{lu}(g_{ij})) \\ &+ \frac{1}{2}\sum_{l=1}^{r}\sum_{u=1}^{r} g^{kl}(C^u_{ij}g_{ul} + C^u_{li}g_{uj} + C^u_{lj}g_{ui}) \end{aligned}$$

where the structures functions $b^{si}, C^u_{st} \in C^\infty(U)$ are given by

$$\sharp(a_s) = \sum_{i=1}^{n} b^{si}\partial_{x_i} \quad (s = 1, ..., r)$$

and

$$[a_s, a_t] = \sum_{u=1}^{r} C^u_{st} a_u \quad (s, t = 1, ..., r),$$

$g_{ij} =< a_i, a_j >$ and (g^{ij}) denote the inverse matrix of (g_{ij}).

3 A-geodesic Curves

Definition 3.1. *Let* $p : A \to M$ *be a Lie algebroid with a linear A-connection* D. *An A-geodesic is an A-path* α *which satisfy :*

$$D^\alpha \alpha = 0$$

In local coordinate this A-geodesic are characterized by differential equations as shown by the following proposition.

Proposition 3.1. *[1] Let $(x_1, \dots, x_n)$ be a locals coordinates system on an open subset U of M and $\{a_1, \dots, a_r\}$ a local base of sections on U. An A-path is an A-geodesic if : for all $i = 1, \dots, n$ and $j = 1, \dots, r$ one has:*

$$\dot{x_i}(t) = \sum_{j=1}^{r} \alpha(t) b^{ji}(x_1(t), \cdots, x_n(t)), \tag{6}$$

$$\dot{\alpha_j}(t) = -\sum_{s,u=1}^{r} \alpha_s(t)\alpha_u(t)\Gamma^j_{su}(x_1(t), \cdots, x_n(t)); \tag{7}$$

where $\alpha(t) = \sum_{i=1}^{r} \alpha_i(t) a_i$ is the local expression of α and $p(\alpha(t)) = (x_1(t), \cdots, x_n(t))$ is the local expression of the base path.

Remark 3.1. *Note here that for all $x \in M$ and $a_0 \in A_x$ there is an unique A-geodesic α such that $\alpha(0) = a_0$ and $p(\alpha(0)) = x$.*

Moreover, as consequence of the definition 2.4, the following theorem can be mentioned

Theorem 3.1. *Let $p : A \to M$ be a Lie algebroid with anchor map $\sharp$. Let D be a linear A-connection compatible with Lie algebroid structure. If α is an A-geodesic, then her base path is a TM-geodesic. Moreover if $\sharp$ is injective then for all $x \in M$ there is an A-geodesic α such that $p(\alpha(0)) = x$.*

Proof. Let α be an A-geodesic with base path γ. Then we have:

$$D^\alpha \alpha = 0 \Rightarrow \sharp(D^\alpha \alpha) = 0$$

Since $\sharp D = \nabla \sharp$ and $\sharp(D^\alpha \alpha) = \nabla_{\sharp\alpha} \sharp\alpha$ one has :

$$\nabla_{\dot{\gamma}} \dot{\gamma} = \nabla_{\sharp\alpha} \sharp\alpha = 0.$$

For all $x \in M$ and $X \in T_x M$ there is a TM-geodesic γ such that $\gamma(0) = x$ and $\dot{\gamma} = X$. Since p is a surjection there is an A-path α such that $\frac{d}{dt} p(\alpha(t)) = \dot{\gamma}$. Hence $\sharp(\alpha) = \dot{\gamma}$
thus

$$\nabla_{\dot{\gamma}} \dot{\gamma} = \nabla_{\sharp\alpha} \sharp\alpha = \sharp D^\alpha \alpha = 0$$

As $\sharp$ is injective one has $D^\alpha \alpha = 0$.

□

4 Geodesically Complete Lie Algebroid

As in the classic case, we will call maximal A-geodesic, an A-geodesic which is defined on all $\mathbb{R}$. The existence of this notion can be found with an application of the theorem 3.1 in the case of maximal A-geodesic. Hence one has the following lemma.

Lemma 4.1. *Let $p : A \to M$ be a Lie algebroid with anchor map $\sharp$. If $\sharp$ is injective then for all $x \in M$ there is a maximal A-geodesic α such that $p(\alpha(0)) = x$.*

Proof. For all $x \in M$ and $X_x \in T_x M$ there is a maximal TM-geodesic γ such that $\gamma(0) = x$ and $\dot{\gamma} = X_x$. As from the proposition 3.1 there is an A-geodesic α such that γ is the base path and $\alpha(0) = a_0$. Since γ is maximal then α is also maximal. □

Remark 4.1. *Any maximal A-geodesic induces a maximal TM-geodesic*

Thus, we can set the definition of a geodesically complete Lie algebroid

Definition 4.1. *A Lie algebroid $p : A \to M$ is said to be geodesically complete if any A-geodesic is maximal.*

Theorem 4.1. *Let $p : A \to M$ be a Lie algebroid with anchor map $\sharp$. If the anchor is injective, then the following assertions are equivalent:*

1) A is geodesically complete;

2) M is goedesictally complete.

Proof. 1) $\Rightarrow$ 2). For $x \in M, a_0 \in A_x$. Let γ be a TM-geodesic such that $\gamma(0) = x$, from the lemma there is an A-geodesic α such that $\alpha(0) = a_0$ and with base path γ. Since α is defined in $\mathbb{R}$, one has γ defined in $\mathbb{R}$.
2) $\Rightarrow$ 1). For all $x \in M$ and $a_0 \in A_x$. Let α be an A-geodesic such that $\alpha(0) = a_0$. Here base path γ is a TM-geodesic. □

5 Riemannian Distance

Let $p : A \to M$ be a Lie algebroid with anchor map $\sharp$ and g be a Riemannian metric on A. Suppose M be a connected manifold. We denote by Ω_{xy} the set of smooth path $\gamma : [0, 1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = 1$; and by $\tilde{\Omega}_{xy}$ the set of A-path α with base path $\gamma \in \Omega_{xy}$.

Proposition 5.1. *For any $x, y \in M$ the set $\tilde{\Omega}_{x,y}$ of A-path with end points x and y is not empty.*

Proof. For any $x, y \in M$ the set Ω_{xy} of paths on M is not empty. Either for any path, $\gamma \in \Omega_{xy}$ there is an A-path α such that γ is the base path of α, then $\alpha \in \tilde{\Omega}_{xy}$. □

The most important fact is that we can compute the length of any A-path $\alpha \in \Omega_{xy}$ like in the classic case of Riemannian manifold. Which give the following definition.

Definition 5.1. *Let $\alpha : [0, 1] \to A$ be an A-connection with base path γ such that $p(\alpha(0)) = x$ and $p(\alpha(1) = y$, then the length $\mathcal{L}(\alpha)$ of α is given by:*

$$\mathcal{L}(\alpha) = \int_0^1 (g(\alpha(t), \alpha(t))^{\frac{1}{2}} dt \tag{8}$$

Now, we can set.

$$d(x, y) = inf\{\mathcal{L}(\alpha), \alpha \in \tilde{\Omega}_{xy}\} \tag{9}$$

Then we have the following proposition.

Proposition 5.2. *d is a distance on M, ie:*

1. *$d(x, y) \geq 0$ with equality if $x = y$;*
2. *$d(x, y) = d(y, x)$;*
3. *$d(x, y) \leq d(x, z) + d(z, y)$.*

Proof. 1. Supposed $d(x, y) = 0$ with $x \neq y$. Let U_x be an open neighborhood of $x \in M$. Then $\forall \epsilon > 0$ there is an A-path $\alpha \in \tilde{\Omega}_{xy}$ such that $\mathcal{L}(\alpha) < \epsilon$
thus $\forall n \in \mathbb{N}$ there is $\alpha_n \in \tilde{\Omega}_{xy}$ such that $\mathcal{L}(\alpha_n) < \frac{1}{n}$
then we have

$$\lim_{n\to\infty} \mathcal{L}(\alpha_n) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \int_0^1 (g(\alpha_n, \alpha_n))^{\frac{1}{2}} dt = 0$$

with the continuty of the integrale and the Riemannian metric one has

$$\begin{aligned} |\alpha| = \lim_{n\to\infty} |\alpha_n| = 0 \quad &\Rightarrow \quad \sharp\alpha = 0 \\ &\Rightarrow \quad \dot{\gamma} = 0 \\ &\Rightarrow \quad \gamma = \text{ constant} \end{aligned}$$

then $x = y$. Contradiction.

2. It's easy to see here as in the classic case of Riemannian manifold, that if α is an A-path with base path γ and $\phi : [t_0, t_1] \to [t_0, t_1], t \mapsto t_0 + t_1 - t$ then the A-path $\alpha \circ \phi$ with base path $\gamma \circ \phi$ is the inverse A-path of α. Moreover, α and $\alpha \circ \phi$ have the same length. And, if α is in Ω_{xy} then $\alpha \circ \phi$ is in Ω_{yx}.
3. Let $\alpha_1 : [t_0, t_1] \to A$ and $\alpha_2 : [t_1, t_2] \to A$ be two A-path. Then the union A- path $\alpha : [t_0, t_2] \to A$ of α_1 and α_2 ($\alpha = \alpha_1 \cup \alpha_2$) is such that $\mathcal{L}(\alpha) = \mathcal{L}(\alpha_1) + \mathcal{L}(\alpha_2)$. Suppose that $\alpha_1 \in \Omega_{xz}$ and $\alpha_2 \in \Omega_{zy}$, then $\alpha \in \Omega_{xy}$. Hence, one has $d(x, y) \leq \mathcal{L}(\alpha_1) + \mathcal{L}(\alpha_2)$. With the infimum, one has the inequality.

□

Now, let $\sharp$ be surjective. For any $x \in M$, we denote by $\mathcal{G}_x$ the kernel of $\sharp_x$. Since g is non-degenerate, one has :

$$A_x = \mathcal{G}_x \oplus \mathcal{G}_x^{\perp}.$$

Then we have the following proposition.

Proposition 5.1. *The restriction of $\sharp_x$ to $\mathcal{G}_x^{\perp}$ is an isomorphism into T_xM.*

Proof. Since $\sharp$ is surjective then for all $x \in M$, $\sharp_x$ is surjective. Let $\alpha_x, \beta_x \in \mathcal{G}_x^{\perp}$ such that $\sharp_x(\alpha_x) = \sharp_x(\beta_x)$. Then one has:

$$\sharp_x(\alpha_x) = \sharp_x(\beta_x) \quad \Rightarrow \quad \sharp_x(\alpha_x - \beta_x) = 0 \quad \Rightarrow \quad \alpha_x - \beta_x \in \mathcal{G}_x$$

Since g is non-degenerate, one has $\alpha_x - \beta_x = 0$ and $\alpha_x = \beta_x$. □

Moreover, for any $x \in M$ and any $X_x, Y_x \in T_xM$ let's set:

$$\tilde{g}_x(X_x, Y_x) = g_x(\alpha_x, \beta_x) \tag{10}$$

where $\alpha_x, \beta_x \in \mathcal{G}_x^{\perp}$ such that $\sharp_x(\alpha_x) = X_x$ and $\sharp_x(\beta) = Y_x$. Then we have a scalar product $\tilde{g}_x$ on T_xM. This scalar product give rise to a Riemannian metric $\tilde{g}$ on M. Hence, it's induced Riemannian distance $\tilde{d}$ on M. One of the important fact of this construction is that, we have the following equality:

$$A = \mathcal{G} \oplus \mathcal{G}^{\perp}.$$

As Boucetta gives it in [1].

Proposition 5.2. *For any path γ on M there is an A-path on $\mathcal{G}^{\perp}$ and $\mathcal{L}(\gamma) = \mathcal{L}(\alpha)$.*

Proof. Since $\sharp$ is locally bijective on $\mathcal{G}^{\perp}$ into TM one has

$$\mathcal{L}(\gamma) = \int_a^b (\tilde{g}(\dot{\gamma}(t), \dot{\gamma}(t)))^{\frac{1}{2}} dt = \int_a^b (g(\alpha(t), \alpha(t)))^{\frac{1}{2}} dt = \mathcal{L}(\alpha)$$

□

It's clear that for any A-path α and $\overline{\alpha}$ the restriction of α on $\mathcal{G}^{\perp}$ we have :

$$\mathcal{L}(\overline{\alpha}) \leq \mathcal{L}(\alpha).$$

And we have the following proposition

Proposition 5.3. *Let $p : A \to M$ a transitive Riemannian Lie algebroid with anchor map $\sharp$ and Riemannian metric g. Then, the induces Riemannian distances d and $\tilde{d}$ are equivalent on M.*

Proof. Let $x, y \in M$ and $\alpha \in \Omega_{xy}$ with base path γ. then one has:

$$\begin{aligned} \mathcal{L}(\alpha) &= \int_0^1 \sqrt{g(\alpha,\alpha)}dt \\ &= \int_0^1 \sqrt{g(\alpha^{\perp},\alpha^{\perp}) + g(\alpha^{\top},\alpha^{\top})}dt \end{aligned}$$

where $\alpha = \alpha^{\perp} + \alpha^{\top}$ with $\alpha^{\top} \in \mathcal{G}^{\perp}$ and $\alpha^{\perp} \in \mathcal{G}$. Then

$$\mathcal{L}(\alpha) \geq \int_0^1 \sqrt{g(\alpha^{\top},\alpha^{\top})} = \mathcal{L}(\gamma)$$

With the infimum, one has :

$$d(x,y) \geq \tilde{d}(x,y) \quad (*)$$

In the other hand, one has

$$\begin{aligned} \mathcal{L}(\alpha) &= \int_0^1 \sqrt{g(\alpha^{\perp},\alpha^{\perp}) + g(\alpha^{\top},\alpha^{\top})}dt \\ &\leq \int_0^1 \sqrt{g(\alpha^{\perp},\alpha^{\perp})}dt + \int_0^1 \sqrt{g(\alpha^{\top},\alpha^{\top})}dt \\ &\leq \int_0^1 \sqrt{g(\alpha^{\top},\alpha^{\top})}dt + \lambda \int_0^1 \sqrt{g(\alpha^{\top},\alpha^{\top})}dt \\ &\leq (1+\lambda)\int_0^1 \sqrt{g(\alpha^{\top},\alpha^{\top})}dt \end{aligned}$$

with the infimum, one has

$$d(x,y) \leq (1+\lambda)\tilde{d}(x,y) \quad (**)$$

At last with $(*)$ and $(**)$, one has

$$\tilde{d}(x,y) \leq d(x,y) \leq (1+\lambda)\tilde{d}(x,y)$$

□

Theorem 5.1. *Let $p : A \to M$ be a geodesically complete Riemannian and transitive Lie algebroid with anchor map $\sharp$ and Riemannian metric g. If M is connected, them it's a complete metric space. Hence, any closed and bounded subset of M is compact.*

Proof. Since A is geodesically complete then with the remark 4.1 and the proposition 5.1, one can say that M is geodesically complete. From the above construction, there is an induced Riemannian metric $\tilde{g}$ and an induced Riemannian distance $\tilde{d}$. With the Hopf Rinow's theorem we have the following equivalent assertions

1. M is geodesically complete;

2. $(M, \tilde{d})$ is complete;
3. any bounded and closed subset of M is compacte.

□

The following corollary is a consequense of theorem 5.1. It gives a part of the Hopf Rinow theorem on a Riemannian transitive Lie algebroid with base manifold connected.

Corollary 5.1. *Let $p : A \to M$ be a Riemannian Lie algebroid with anchor map $\sharp$ and Riemannian metric g. If $\sharp$ is injective and M is a connected manifold, then the following assertions are equivalent :*

1. *(A, g) is geodesically complete.*
2. *(M, d) is complete.*
3. *Any closed and bounded subset of M is compact.*

Proof. By using theorem 4.1, one has 1) $\Leftrightarrow$ 2) and the Hopf Rinow theorem gives the last equivalence. □

As an application of the theorem 5.1 we have the following corollary. It's also a consequence of this theorem.

Corollary 5.2. *Let $p : A \to M$ be a Riemannian geodesically complete Lie algebroid with anchor map $\sharp$. Any connected, bounded and closed Leaf of a characteristic foliation is compact and complete. Moreover, if L is connected, then any closed and bounded subset of L is compact.*

Proof. Any leaf L of a characteristic foliation of a Lie algebroid induces a transitive Lie algebroid $p_L : A_L \to L$ which anchor map is the restriction of the Lie algebroid's anchor to A_L. We conclude with the theorem 5.1. □

6 Example

1. If $A = TM$ then, we have the tangent Lie algebroid. The metric g is a Riemannian metric on the manifold M. Moreover, if M is connected then we have, for all A-path α with base path γ, $\sharp\alpha = \dot{\gamma}$. As $\sharp = id_{TM}$, then $\alpha = \dot{\gamma}$. Hence
$$\mathcal{L}(\alpha) = \int_0^1 \sqrt{g(\alpha(t), \alpha(t))}dt = \int_0^1 \sqrt{g(\dot{\gamma}, \dot{\gamma})}dt = \mathcal{L}(\gamma)$$
then $d = \tilde{d}$
It's clear that the geodesically complete structure of A is the natural geodesically complete structure of the Riemannian manifold (M, g).
2. Let (M, ω) be a symplectic manifold, then there is a Lie algebroid structure on T^*M induced by:
   - a Lie bracket of differential 1-form on $\Gamma(T^*M)$ defined by the isomorphism $\tilde{\Pi} = \tilde{\omega}^{-1} : T^*M \to TM$ such that $\tilde{\omega}(u) = \omega(u, \cdot)$.
   - the anchor map $\sharp = -\tilde{\Pi}$.

The structure is called symplectic Lie algebroid structure (see [12] for more details). For any A-path α, with base path γ, one has $\sharp\alpha = \dot{\gamma}$. Since Π is an isometry, then the metrics g and $\tilde{g}$ are related by $g(\alpha, \beta) = \tilde{g}(\sharp\alpha, \sharp\beta)$ and we have:

$$\begin{aligned}
\mathcal{L}(\alpha) &= \int_0^1 \sqrt{g(\alpha(t), \alpha(t))}dt \\
&= \int_0^1 \sqrt{\tilde{g}(\sharp\alpha(t), \sharp\alpha(t))}dt \\
&= \int_0^1 \sqrt{\tilde{g}(\dot{\gamma}(t), \dot{\gamma}(t))}dt \\
&= \mathcal{L}(\gamma)
\end{aligned}$$

and $d = \tilde{d}$.
The geodesically complete structure of A is equivalent to the natural geodesically complete structure of the Riemannian manifold.

3. If $A = T^*M$, and M is a Poisson manifold with Poisson bivector π, then we have the Lie algebroid of the Poisson manifold $(T^*M, [,]_\pi, \sharp)$ [10]. Moreover, if $\tilde{g}$ is a Riemannian metric on M, then there is a natrural isometry, $\sharp_{\tilde{g}} : T^*M \to TM$, which generalises this metric to T^*M by : $g(\alpha, \beta) = \tilde{g}(\sharp_{\tilde{g}}(\alpha), \sharp_{\tilde{g}}(\beta))$ [13] and for all A-path α, with base path γ, we have $\sharp\alpha = \dot{\gamma}$ and $\sharp_{\tilde{g}}\alpha = \dot{\gamma}$. Then,

$$\begin{aligned}
\mathcal{L}(\alpha) &= \int_0^1 \sqrt{g(\alpha(t), \alpha(t))}dt \\
&= \int_0^1 \sqrt{\tilde{g}(\sharp_{\tilde{g}}\alpha(t), \sharp_{\tilde{g}}\alpha(t)}dt \\
&= \int_0^1 \sqrt{\tilde{g}(\dot{\gamma}(t), \dot{\gamma}(t))}dt \\
&= \mathcal{L}(\gamma).
\end{aligned}$$

Thus, we have $d = \tilde{d}$. Therefore, there is a natural equivalence between the geodesically complete structure of A and the geodesically complete structure of M.

Competing Interests

Authors have declared that no competing interests exist.

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