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Hyers-Ulam-Rassias Instability for Linear and Nonlinear Systems of Differential Equations

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

This paper considers Hyers-Ulam-Rassias instability for linear and nonlinear systems of differential equations. Integral sufficient conditions of Hyers-Ulam-Rassias instability and Hyers-Ulam instability for linear and nonlinear systems of differential equations are established. Illustrative examples will be given.

Keywords: Hyers-Ulam-Rassias instability; nonlinear systems; differential equations.

1 Introduction

In 1940, Ulam [1] posed the stability problem of functional equations. In the talk, Ulam discussed a problem concerning the stability of homomorphisms. A significant breakthrough came in 1941, when Hyers [2] gave a partial solution to Ulam's problem. During the last two decades very important contributions to the stability problems of functional equations were given by many mathematicians [3-7]. More than twenty years ago, a generalization of Ulam's problem was proposed by replacing functional equations with differential equations: The differential equation $F(t, u(t), u'(t), ..., u^{(n)}(t)) = 0$ has the Hyers-Ulam stability if for given $\varepsilon > 0$ and a function y such that

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$$\left|F(t,u(t),u'(t),\ldots,u^{(n)}(t))\right| \le \varepsilon$$

there exists a solution y_0 of the differential equation such that

$$| u(t) - u_0(t) | \le K(\varepsilon) \text{ and } \lim_{\varepsilon \to 0} K(\varepsilon) = 0.$$

The first step in the direction of investigating the Hyers-Ulam stability of differential equations was taken by Oblosa in [8,9]. Thereafter, Alsina and Ger [10] have studied the Hyers-Ulam stability of the linear differential equation $z'(t) = z(t)$. The Hyers-Ulam stability problems of linear differential equations of first order and second order with constant coefficients were studied in the papers [11-14]. The results given in [11-14] have been generalized by Popa and Rus [15,16] for the linear differential equations of nth order with constant coefficients.

In addition to above-mentioned studies, some authors have studied the Hyers-Ulam stability for systems of differential equations [17,18].

In [19] Brillouët-Belluot indicated that there are only few outcomes of which we could say that they concern nonstability of functional equations. However in [20] Qarawani investigate the Hyers-Ulam instability of linear $z'' + z = \alpha(x)z$ and nonlinear $z'' + z = h(x)z^{\beta}$ differential equations of second order.

The objective of this article is to investigate the Hyers-Ulam-Rassias instability for autonomous linear system of differential equations

$$z' = Cz + f(z) \tag{1.1}$$

and for a linear system with almost constant matrix

$$z' = (C + D(t))\, z \tag{1.2}$$

Moreover this paper considers the Hyers-Ulam-Rassias instability for the non-linear system

$$z' = Cz + h(t,z) \tag{1.3}$$

with the initial condition

$$z(0) = 0 \tag{1.4}$$

where C is a constant $n \times n$ matrix, $\mathrm{f} = (\mathrm{f}_1,\ldots,f_n)^T$ are continuously differentiable n-dimensional column vector-functions such that $f(0) = 0, 0 = (0,\ldots,0)^T \in \mathbb{R}^n$, $D(t)$ is a matrix valued on the interval function I, such that $\int_0^\infty \|D(t)\| dt < \infty$ and $h(t,z)$ is continuous in $I \times \mathbb{R}^n \to \mathbb{R}^n$ such that for sufficiently large t

$$\|h(t,z)\| \sim \gamma(t)\|z(t)\|^{\alpha}, \tag{1.5}$$

where $\gamma(t)$ is a positive bounded function on $I, h(t,0) = 0, \alpha \in [0,\infty)$ and $I = [0,\infty)$.

It should be noted that Euclidean n-space $\mathbb{R}^n, n \geq 1$, is equipped with the distance

$$\| u - v \| = \left[\sum_{i=1}^{n} (|u_i - v_i|)^2 \right]^{\frac{1}{2}}$$

unless otherwise stated.

2 Preliminaries

Here some definitions are introduced as follows:

Definition 2.1 [13] Let $\phi(t) = col(\phi_1(t), \phi_2(t), ..., \phi_n(t))$ and $\varphi(t) = \|\phi(t)\|$ such that $\phi_i : I \to [0, \infty), i = 1, 2, ..., n$. Equation (1.1) has the Hyers-Ulam-Rassias (HUR) stability with respect to $\varphi : I \to [0, \infty)$ if there exists a positive constant $k > 0$ with the following property: For each $z(t) \in C^1(I, \mathbb{R}^n)$, if

$$\left| z' - Cz - f(z) \right| \leq \phi(t), \tag{2.1}$$

then there exists some $z_0(t) \in C^1(I, \mathbb{R}^n)$ of the equation (1.1) such that

$$\| z(t) - z_0(t) \| \leq k\varphi(t)\ , t \in [0, \infty) \tag{2.2}$$

Definition 2.2 [13] Suppose that $\phi(t) = col(\phi_1(t), \phi_2(t), ..., \phi_n(t))$ and $\varphi(t) = \|\phi(t)\|$ such that $\phi_i : I \to [0, \infty), i = 1, 2, ..., n$. Equation (1.2) has the Hyers-Ulam-Rassias (HUR) stability with respect to $\varphi : I \to [0, \infty)$ if there exists a positive constant $k > 0$ with the following property: For each $z(t) \in C^1(I, \mathbb{R}^n)$, if

$$\left| z' - Cz - h(t, z) \right| \leq \phi(t), \tag{2.3}$$

then there exists some $z_0(t) \in C^1(I, \mathbb{R}^n)$ of the equation (1.3) such that

$$\| z(t) - z_0(t) \| \leq k\varphi(t)\ \ , t \in I \tag{2.4}$$

Lemma2.1 [21] Let C(t) be an $n \times n$ matrix whose elements are functions of parameter t. If

$$C(t). \int_0^t C(r)dr = \int_0^t C(r)dr. C(t),$$

then the chain rule satisfies

$$\frac{d}{dt}\left(\exp\int_0^t C(r)dr\right) = C(t)\exp\int_0^t C(r)dr$$

3 Main Results on Hyers-Ulam-Rassias Instability

Theorem 3.1 Let C be a constant $n \times n$ matrix, $f(z) \in \mathbb{R}^n$ be a continuous vector column in the interval I . Suppose $z(t)$ satisfies the inequality (2.1) with initial condition (1.4) on the interval $0 \le t \le T \le \infty$ and $\varphi(t) : [0, \infty) \to (0, \infty)$ is a continuous function such that the following integral converges:

$$\int_0^t \left\| e^{-C(s-t)} \right\| \varphi(s) ds < \infty, \quad \forall t \ge 0. \tag{3.1}$$

If $\left\| \int_0^t e^{C(t-s)} ds \right\| \to \infty$, as $t \to \infty$ then the zero solution of (1.1) is unstable in the sense of Hyers-Ulam-Rassias.

Proof. Let $z(t)$ be an approximate solution of the initial value problem (1.1),(1.4). It must be shown that zero solution $z_0(t) \equiv 0$ of problem (1.1-1.3) will satisfy the inequality

$$\| z(t) - z_0(t) \| > k\varphi(t)$$

On the contrary, let us assume that there exists $\varphi(t) > 0$ such that $\sup\limits_{t \ge 0} | z(t) - z_0(t) | \le k\varphi(t)$. Then it can be found a constant $M > 0$ such that $M = \sup\limits_{t \ge 0} \| z(t) \|$.

Inequality (2.1) implies that

$$-\phi(t) \le z' - Cz - f(z) \le \phi(t). \tag{3.2}$$

Left-multiplying the inequality (3.2) factor e^{-Ct} yields

$$-e^{-Ct}\phi(t) \le \frac{d}{dt}\left(e^{-Ct} z\right) - e^{-Ct} f(z) \le e^{-Ct}\phi(t). \tag{3.3}$$

Integrating (3.3) from 0 to t gives

$$-\int_0^t e^{-Cs}\phi(s)ds \le e^{-Ct} z - \int_0^t e^{-Cs} f(z) ds \le \int_0^t e^{-Cs}\phi(s)ds,$$

or, equivalently

$$-\int_0^t e^{C(t-s)}\phi(s)ds \le z - \int_0^t e^{C(t-s)} f(z)ds \le \int_0^t e^{C(t-s)}\phi(s)ds.$$

Therefore the following estimate holds true

$$z - \int_0^t e^{C(t-s)} f(z)ds \ge -\int_0^t e^{C(t-s)}\phi(s)ds,$$

or, equivalently

$$z(t) \ge -\int_0^t e^{C(t-s)}\phi(s)ds + \int_0^t e^{C(t-s)} f(s)ds.$$

Consequently

$$\begin{aligned}\|z(t)\| &\ge \left\|\int_0^t e^{C(t-s)} f(s)ds\right\| - \int_0^t \left\|e^{C(t-s)}\right\| |\phi(s)|\, ds \\ &\ge f(z(s^*))\left\|\int_0^t e^{C(t-s)}ds\right\| - \int_0^\infty \left\|e^{-C(s-t)}\right\| \varphi(s)ds,\ \forall t \ge 0.\end{aligned}$$

Then (3.1) and $\left\|\int_0^t e^{C(t-s)}ds\right\| \to \infty,\ t \to \infty$ implies that $\sup_{t\ge 0}\|z(t)\|$ is infinite. Contradiction, which completes the proof of Theorem 3.1.

Corollary 3.1 Replacing $\phi(t)$ by $\varepsilon = col(\varepsilon_1,\ldots,\varepsilon_n)$ in the inequality (3.2) one can get Hyers-Ulam instability for Eq. (1.1) in the interval $0 \le t_0 \le t \le T$, i.e. if $x(t)$ satisfies

$$|z' - Cz - f(z)| \le \varepsilon,$$

with the initial zero condition, then the zero solution $z_0(t) \equiv 0$ of the equation (1.1) satisfies

$\sup\|z(t) - z_0(t)\| \to \infty$, as $t \to \infty$.

The proof of Corollary 3.1 is quite similar to the proof of Theorem 3.1 and will therefore be omitted.

Example 3.1 Consider the non-homogeneous system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = y + 3xy \\ \dfrac{dy}{dt} = -2y - x + 2x^2 \end{cases} \tag{3.4}$$

with initial condition $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

or

$$\frac{dz}{dt} = Cz + f(z)$$

where

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \ C = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, f(z) = \begin{pmatrix} 3xy \\ 2x^2 \end{pmatrix}.$$

On the contrary, assume that there exists $\varphi(t) > 0$ such that $\sup\limits_{t \geq 0} |z(t)| \leq k\varphi(t)$. Then a constant $M > 0$ can be found such that

$$M = \sup_{t \geq 0} \| z(t) \|.$$

One can find a matrix e^{Ct} such that

$$e^{Ct} = \begin{pmatrix} \mathrm{e}^{-t} + t\mathrm{e}^{-t} & t\mathrm{e}^{-t} \\ -t\mathrm{e}^{-t} & \mathrm{e}^{-t} - t\mathrm{e}^{-t} \end{pmatrix}$$

Suppose that

$$\left| \frac{dz}{dt} - Cz - f(x) \right| \leq \varphi(t).$$

From which it follows that

$$-\phi(t) \leq z' - Cz - f(x) \leq \phi(t), \tag{3.5}$$

where $\phi(t) = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}$ and $\varphi(t) = \|\phi(t)\| = \sqrt{\left(e^{-t}\right)^2 + \left(e^{-t}\right)^2} = \sqrt{2}e^{-t}$.

Multiply the inequality (3.5) on the left, by a matrix integrating factor

$$e^{-Ct} = \begin{pmatrix} \mathrm{e^t\text{-}te^t} & -\,\mathrm{te^t} \\ \mathrm{te^t} & \mathrm{e^t+te^t} \end{pmatrix},$$

to get

$$\begin{aligned} &-\begin{pmatrix} \mathrm{e^t\text{-}te^t} & -\,\mathrm{te^t} \\ \mathrm{te^t} & \mathrm{e^t+te^t} \end{pmatrix}\begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} \\ &\leq \frac{d}{dt}\left(\begin{pmatrix} \mathrm{e^t\text{-}te^t} & -\,\mathrm{te^t} \\ \mathrm{te^t} & \mathrm{e^t+te^t} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right) - \begin{pmatrix} \mathrm{e^t\text{-}te^t} & -\,\mathrm{te^t} \\ \mathrm{te^t} & \mathrm{e^t+te^t} \end{pmatrix}\begin{pmatrix} 0 \\ t \end{pmatrix} \\ &\leq \begin{pmatrix} \mathrm{e^t\text{-}te^t} & -\,\mathrm{te^t} \\ \mathrm{te^t} & \mathrm{e^t+te^t} \end{pmatrix}\begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}, \end{aligned} \tag{3.6}$$

and integrating (3.6) from 0 to t yields

$$\begin{aligned} -\begin{pmatrix} t - t^2 \\ t + t^2 \end{pmatrix} &\leq \begin{pmatrix} \mathrm{e^t\text{-}te^t} & -\,\mathrm{te^t} \\ \mathrm{te^t} & \mathrm{e^t+te^t} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \int_0^t \begin{pmatrix} \mathrm{e^s\text{-}se^s} & -\,\mathrm{se^s} \\ \mathrm{se^s} & \mathrm{e^s+se^s} \end{pmatrix}\begin{pmatrix} 3xy \\ 2x^2 \end{pmatrix} ds \\ &\leq \begin{pmatrix} t - t^2 \\ t + t^2 \end{pmatrix}. \end{aligned}$$

By multiplying the last inequality on the left by

$$e^{Ct} = \begin{pmatrix} \mathrm{e^{-t}+te^{-t}} & \mathrm{te^{-t}} \\ \mathrm{-te^{-t}} & \mathrm{e^{-t}\text{-}te^{-t}} \end{pmatrix}$$

yields the inequality

$$\begin{aligned} -\begin{pmatrix} te^{-t} + t^2e^{-t} \\ te^{-t} - t^2e^{-t} \end{pmatrix} &\leq \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 3x(s^*)y(s^*) \\ 2x^2(s^*) \end{pmatrix} \int_0^t \mathrm{e^{-(t\text{-}s)}} \begin{pmatrix} \mathrm{1+t\text{-}s} & t\text{-s} \\ \mathrm{s\text{-}t} & \mathrm{1\text{-}t+s} \end{pmatrix}\begin{pmatrix} 3xy \\ 2x^2 \end{pmatrix} ds \\ &\leq \begin{pmatrix} te^{-t} + t^2e^{-t} \\ te^{-t} - t^2e^{-t} \end{pmatrix} \end{aligned}$$

.

Consequently

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &\geq \begin{pmatrix} 3x(s^*)y(s^*) \\ 2x^2(s^*) \end{pmatrix}\begin{pmatrix} t + 2e^{-t} + te^{-t} - 2 \\ 1 - e^{-t} - te^{-t} \end{pmatrix} - \begin{pmatrix} te^{-t} + t^2e^{-t} \\ te^{-t} - t^2e^{-t} \end{pmatrix} \\ &= \begin{pmatrix} 3x(s^*)y(s^*) \\ 2x^2(s^*) \end{pmatrix}\begin{pmatrix} t + 2e^{-t} - t^2e^{-t} - 2 \\ 1 - e^{-t} - 2te^{-t} + t^2e^{-t} \end{pmatrix} \end{aligned}$$

Thus as $t \to \infty, \sup_{t\geq 0}\|z(t)\|$ becomes infinite. The contradiction establishes the HUR instability of non-homogeneous system (3.4).

Theorem 3.2 Let A be a constant matrix and let $D(t)$ be a continuous matrix valued function in the interval I such that $\int_0^\infty \|D(s)\|ds < \infty$. Suppose $\varphi(t) : [0,\infty) \to (0,\infty)$ is a continuous function such that

$$\int_0^t \left\|e^{-C(s-t)}\right\|\varphi(s)ds < \infty, \quad \forall t \geq 0. \tag{3.9}$$

If the inequality satisfies

$$\left|z' - Cz - D(t)z\right| \leq \phi(t) \tag{3.10}$$

and the integral

$$\left\|\int_0^t e^{C(t-s)}D(s)ds\right\| \to \infty,\ t \to \infty, \tag{3.11}$$

then the solution of (1.2) is unstable in the sense of Hyers-Ulam-Rassias.

Here the norm $\|D(t)\|$ denotes the sum of the absolute values of elements of the matrix $D(t)$.

Proof. Let $z\,(t)$ be an approximate solution of the initial value problem (1.1),(1.3). It will be shown that zero solution $z_0(t) \equiv 0$ of problem (1.1-1.3) will satisfy the inequality

$$\|z(t) - z_0(t)\| > k\varphi(t)$$

On the contrary, assume that there exists $\varphi(t) > 0$ such that $\|z(t)\| \leq k\varphi(t)$. That is, $z(t)$ must be bounded for any $t > 0$.

The inequality (3.10) implies that

$$-\phi(t) \leq z' - Cz - D(t)z \leq \phi(t). \tag{3.12}$$

Multiplying (3.12) on the left, by a matrix integrating factor e^{-At} yields

$$-e^{-Ct}\phi(t) \leq \frac{d}{dt}\left(e^{-Ct}z\right) - e^{-Ct}D(t)z \leq e^{-Ct}\phi(t). \tag{3.13}$$

Integrating (3.13) from 0 to t, implies that

$$-\int_0^t e^{-Cs}\phi(s)ds \le e^{-Ct}z - \int_0^t e^{-Cs}D(s)z(s)ds \le \int_0^t e^{-Cs}\phi(s)ds,$$

or, equivalently

$$-\int_0^t e^{C(t-s)}\phi(s)ds \le z - \int_0^t e^{C(t-s)}D(s)z(s)ds \le \int_0^t e^{C(t-s)}\phi(s)ds. \tag{3.14}$$

Inequality (3.14) implies an estimate

$$z - \int_0^t e^{C(t-s)}D(s)z(s)ds \ge -\int_0^t e^{C(t-s)}\phi(s)ds,$$

or $$z(t) \ge -\int_0^t e^{C(t-s)}\phi(s)ds + \int_0^t e^{C(t-s)}D(s)z(s)ds.$$

Use the Mean Value Theorem for integrals to obtain

$$\begin{aligned}\|z(t)\| &\ge \left\|\int_0^t e^{C(t-s)}D(s)z(s)ds\right\| - \int_0^t \left\|e^{C(t-s)}\right\| |\phi(s)|\, ds \\ &\ge \|z(s^*)\| \left\|\int_0^t e^{C(t-s)}D(s)ds\right\| - \int_0^\infty \left\|e^{-C(s-t)}\right\| \varphi(s)ds,\ \forall t \ge 0.\end{aligned}$$

From (3.9) and (3.11), it implies that $\sup_{t\ge 0}\|z(t)\|$ is infinite. The contradiction completes the proof of Theorem 3.2.

Example 3.2 Consider the system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = e^{-t}x, \end{cases} \tag{3.15}$$

with initial condition $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

Or, equivalently

$$\frac{dz}{dt} = Cz + D(t)z$$
,

where

$$z=\begin{pmatrix}x\\y\end{pmatrix},\ C=\begin{pmatrix}0&1\\0&0\end{pmatrix},\ D(t)=\begin{pmatrix}0&0\\e^{-t}&0\end{pmatrix}.$$

One can find

$$e^{Ct}=\begin{pmatrix}1&t\\0&1\end{pmatrix}$$

Let $z(t)$ be an approximate solution of the zero initial value problem (3.15). It will be shown that zero solution $z_0(t)\equiv 0$ of problem (3.15) will satisfy the inequality

$$\|z(t)-z_0(t)\|>k\varphi(t)$$

On the contrary, let us assume that there exists $\varphi(t)>0$ such that $\|z(t)\|\leq k\varphi(t)$, and hence $z(t)$ must be bounded for any $t>0.$

Suppose that

$$\left|\frac{dz}{dt}-Cz-D(t)z\right|\leq\phi(t).$$

From which it follows that

$$-\phi(t)\leq z'-Cz-D(t)z\leq\phi(t), \tag{3.16}$$

where $\phi(t)=\begin{pmatrix}e^{-t}\\0\end{pmatrix}$ and $\varphi(t)=\|\phi(t)\|=\sqrt{0+\left(e^{-t}\right)^2}=e^{-t}.$

Multiply the inequality (3.16) on the left, by a matrix integrating factor

$$e^{-Ct}=\begin{pmatrix}1&-t\\0&1\end{pmatrix},$$

to get

$$-\begin{pmatrix}e^{-t}\\0\end{pmatrix}\leq\frac{d}{dt}\left(\begin{pmatrix}1&-t\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\right)-\begin{pmatrix}0&0\\e^{-t}&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\leq\begin{pmatrix}e^{-t}\\0\end{pmatrix}, \tag{3.17}$$

and integrating (3.17) from 0 to t gives

$$-\begin{pmatrix} 1-e^t \\ 0 \end{pmatrix} \le \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \int_0^t \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \mathrm{e}^{-s} & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} ds \le \begin{pmatrix} 1-e^t \\ 0 \end{pmatrix}.$$

By multiplying the last inequality on the left by

$$e^{Ct} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

it follows that

$$-\begin{pmatrix} 1-e^t \\ 0 \end{pmatrix} \le \begin{pmatrix} x \\ y \end{pmatrix} - \int_0^t \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \mathrm{e}^{-s} & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} ds$$
$$\le \begin{pmatrix} 1-e^t \\ 0 \end{pmatrix}$$

.

By applying the Mean Value Theorem to the integral above, there exists $0 < s^* < t$ such that

$$\begin{pmatrix} x \\ y \end{pmatrix} \ge \begin{pmatrix} t+e^{-t}-1 & 0 \\ 1-e^{-t} & 0 \end{pmatrix}\begin{pmatrix} x(s^*) \\ y(s^*) \end{pmatrix} - \begin{pmatrix} 1-e^t \\ 0 \end{pmatrix}$$

Thus as $t \to \infty$, $\sup\limits_{t \ge 0}\|z(t)\|$ becomes infinite. The contradiction establishes the HUR instability of system (3.15).

Corollary 3.2 Replacing $\phi(t)$ by $\varepsilon = col(\varepsilon_1, \dots, \varepsilon_n)$ in the inequality (3.12) concludes Hyers-Ulam instability for Eq. (1.2) in the interval $0 \le t_0 \le t \le \infty$.

Theorem 3.3 Suppose that $z(t)$ satisfies the inequality (2.3) with initial conditions (1.4). Let C be a constant matrix and $\varphi(t) : I \to (0, \infty)$ be a continuous function such that

$$\int_0^t \left\|e^{-C(s-t)}\right\|\varphi(s)ds < \infty,\ t \to \infty.$$

If (1.5) holds and the integral

$$\left\|\int_0^t e^{C(t-s)}ds\right\| = \infty, t \to \infty, \qquad (3.18)$$

then the solution of (1.3) is instable in the sense of Hyers-Ulam-Rassias as $t \to \infty$.

Proof. Let $z\,(t)$ be an approximate solution of the initial value problem (1.1),(1.3). It will be shown that zero solution $z_0(t) \equiv 0$ of problem (1.1-1.3) will satisfy the inequality

$$\|z(t) - z_0(t)\| > k\varphi(t)$$

On the contrary, let us assume that there exists $\varphi(t) > 0$ such that $\|z(t)\| \leq k\varphi(t), i.e\ z(t)$ is bounded for any $t > 0$.

Then it follows from the inequality (2.3) that

$$-\phi(t) \leq z' - Cz - h(t,z) \leq \phi(t). \tag{3.19}$$

Multiplying (3.19) by matrix integrating factor e^{-At}, gives

$$-e^{-Ct}\phi(t) \leq \frac{d}{dt}\left(e^{-Ct}z\right) - e^{-Ct}h(t,z) \leq e^{-Ct}\phi(t).$$

Integrate the later inequality from 0 to t, to have

$$-\int_0^t e^{C(t-s)}\phi(s)ds \leq z - \int_0^t e^{C(t-s)}h(s,z))ds$$
$$\leq \int_0^t e^{C(t-s)}\phi(s)ds.$$

Now, let us estimate $z(t)$:

$$-\int_0^t e^{C(t-s)}\phi(s)ds \leq z - \int_0^t e^{C(t-s)}h(s,z))ds$$

From which it is obtained

$$z \geq \int_0^t e^{C(t-s)}h(s,z))ds - \int_0^t e^{C(t-s)}\phi(s)ds$$

By virtue of (1.5) and applying the mean value theorem to the first integral, for $0 < s^* < t$ yields

$$\|z(t)\| \geq \left\|\int_0^t e^{C(t-s)}h(s,z)ds\right\| - \int_0^t \left\|e^{C(t-s)}\right\|\varphi(s)ds$$
$$= \gamma(s^*)\|z(s^*)\|^{\alpha}\left\|\int_0^t e^{C(t-s)}ds\right\| - \int_0^t \left\|e^{C(t-s)}\right\|\varphi(s)ds$$

Based on the boundedness assumption, $\|z(s^*)\|^{\alpha}$ will be a constant, and in view of (3.18) $\sup\limits_{t\geq 0}\|z(t)\|$ becomes infinite as $t \to \infty$. The contradiction completes the proof of Theorem 3.3.

Now it will be proved HUR instability for the system (1.3) in which $C(t), t \in [0,\infty)$ is an $n \times n$ matrix of real continuous functions on I. In this case one can similarly get HUR instability for the systems (1.2), (1.3).

Theorem 3.4 Suppose that $z(t)$ satisfies the inequality (2.3) with initial condition (1.4), and C(t) is a continuous matrix function commuting with its integral.

Then a sufficient condition for problem (1.3-1.4) to be instable in the sense of Hyers-Ulam-Rassias as $t \to \infty$ is that

$$(i) \qquad \int_0^t \left\| \exp\left(\int_s^t C(r)dr \right) \right\| \varphi(s)ds < \infty, \;\; t \to \infty$$

$$(ii) \qquad \left\| \int_0^t \exp\left(\int_s^t C(r)dr \right) ds \right\| = \infty, t \to \infty, \tag{3.20}$$

$$(iii) \qquad \|h(t,z)\| \sim \gamma(t)\|z(t)\|^{\alpha}, \tag{3.21}$$

Proof. Let $z(t)$ be an approximate solution of the initial value problem (1.3),(1.4). It will be shown that zero solution $z_0(t) \equiv 0$ of problem (1.1-1.3) will satisfy the inequality

$$\|z(t) - z_0(t)\| > k\varphi(t)$$

On the contrary, suppose there exists $\varphi(t) > 0$ such that $\|z(t)\| \leq k\varphi(t), i.e\ z(t)$ is bounded for any $t > 0.$ Then it follows from the inequality (2.3) that

$$-\phi(t) \leq z' - C(t)z - h(t,z) \leq \phi(t), \tag{3.22}$$

Multiplying (3.22) by matrix integrating factor $\exp\left(-\int_0^t C(r)dr \right)$ gives

$$\begin{aligned} -\mathrm{e}^{-\int_0^t C(r)dr}\phi(t) \leq \mathrm{e}^{-\int_0^t C(r)dr} z' - \mathrm{e}^{-\int_0^t C(r)dr} C(t)z - \mathrm{e}^{-\int_0^t C(r)dr} h(t,z) \\ \leq \mathrm{e}^{-\int_0^t C(r)dr}\phi(t) \end{aligned} \tag{3.23}$$

Since C(t) is a commuting with its integral, then by Lemma 2.1 (3.23) can be written in the following

$$-\mathrm{e}^{-\int_0^t C(r)dr}\phi(t) \le \frac{d}{dt}\left(\mathrm{e}^{-\int_0^t C(r)dr} z\right) - \mathrm{e}^{-\int_0^t C(r)dr} h(t,z) \le \mathrm{e}^{-\int_0^t C(r)dr}\phi(t).$$

Integrate the later inequality from 0 to t, then multiply by matrix integrating factor $\exp\left(\int_0^t C(r)dr\right)$ to have

$$-\mathrm{e}^{\int_0^t C(r)dr}\int_0^t \mathrm{e}^{-\int_0^s C(r)dr}\phi(s)ds \le z - \mathrm{e}^{\int_0^t C(r)dr}\int_0^t \mathrm{e}^{-\int_0^s C(r)dr} h(s,z))ds$$
$$\le \mathrm{e}^{\int_0^t C(r)dr}\int_0^t \mathrm{e}^{-\int_0^s C(r)dr}\phi(s)ds,$$

Collect together the integrals on the left-hand side of inequality to obtain

$$z(t) \ge \int_0^t \mathrm{e}^{\int_s^t C(r)dr} h(s,z))ds - \int_0^t \mathrm{e}^{\int_s^t C(r)dr}\phi(s)ds$$

Use (3.21) and apply the integral mean value theorem to the first integral, to obtain

$$\|z(t)\| \ge \left\|\int_0^t \exp\left(\int_s^t C(r)dr\right) h(s,z)ds\right\| - \int_0^t \left\|\exp\left(\int_s^t C(r)dr\right)\right\|\varphi(s)ds$$
$$= \gamma(s^*)\|z(s^*)\|^\alpha \left\|\int_0^t \exp\left(\int_s^t C(r)dr\right)ds\right\| - \int_0^t \left\|\exp\left(\int_s^t C(r)dr\right)\right\|\varphi(s)ds$$

where $s^* \in [0,t]$. By boundedness assumption on solution $\|z(t)\|$, $\|z(s^*)\|^\alpha$ will be a constant. Therefore, in view of (3.20), it is found that $\sup_{t\ge 0}\|z(t)\|$ becomes infinite as $t \to \infty$. The contradiction completes the proof of Theorem 3.3.

Corollary 3.3 Replacing $\phi(t)$ by $\varepsilon = col(\varepsilon_1,\ldots,\varepsilon_n)$ in inequalities (3.19), (3.22) one can get Hyers-Ulam instability for Eq.(1.3) in the interval $0 \le t_0 \le t \le \infty$, i.e. if $z(t)$ satisfies

$$\left|z' - C(t)z - h(t,z)\right| \le \varepsilon,$$

with the initial condition $z(0) = 0,$, then there exists $z_0(t) \equiv 0$ of the system (1.3) such that

$$\|z(t) - z_0(t)\| \le k\varepsilon\,.$$

The proof of Corollary 3.3 is quite similar to the proof of Theorem 3.4 and will therefore be omitted.

4 Conclusion

Ulam Stability of differential equations is an important subject in the applications. In general terms, the main issue in the Ulam stability is to find a solution of equation differing slightly from a given approximate one, which must be close to solution of the given equation. This is quite useful in many applications, for example Fluid Dynamics, Numerical Analysis, Optimization, Biology, and Economics etc., where finding the exact solution is quite difficult. It also helps if the stochastic effects are small, to use deterministic model to approximate a stochastic one. It is very important to note that there are many other applications for Hyers-Ulam stability in other areas like, nonlinear analysis problems including partial differential equation and integral equations, [22,23].

This paper will encourage researchers to consider instabilities for various differential equations. For the first time, this paper discuss the problem of HUR instability for linear and nonlinear systems of differential equations. Here the indirect method is used to obtain some integral sufficient conditions of HUR instability for linear systems with almost constant matrix, linear and nonlinear systems of differential equations with constant matrix. Moreover in this paper the HUR instability for the nonlinear system with continuous matrix function commuting with its integral is considered . Illustrative examples satisfying the assumptions of the proved theorems are given.

Competing Interests

Author has declared that no competing interests exist.

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