Open Journal of
Discrete Applied Mathematics

PSR Press

Article

S-norms on anti Q-fuzzy subgroups

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Communicated by: Muhammad Kamran Jamil
Received: 27 June 2021; Accepted: 28 August 2021; Published: 6 October 2021.

Abstract: In this paper, by using S-norms, we defined anti fuzzy subgroups and anti fuzzy normal subgroups which are new notions and considered their fundamental properties and also made an attempt to study the characterizations of them. Next we investigated image and pre image of them under group homomorphisms. Finally, we introduced the direct sum of them and proved that direct sum of any family of them is also anti fuzzy subgroups and anti fuzzy normal subgroups under S-norms, respectively.

Keywords: Fuzzy groups; S-norms; Homomorphisms; Direct sum.

MSC: 20N25; 47A30; 20K30; 03E72.

1. Introduction

Most lectures on group theory actually start with the definition of what is a group. It may be worth though spending a few lines to mention how mathematicians came up with such a concept. Around 1770, Lagrange initiated the study of permutations in connection with the study of the solution of equations. He was interested in understanding solutions of polynomials in several variables, and got this idea to study the behaviour of polynomials when their roots are permuted. It is Galois (1811-1832) who is considered by many as the founder of group theory. He was the first to use the term "group" in a technical sense, though to him it meant a collection of permutations closed under multiplication. Galois theory will be discussed much later in these notes.

Fuzzy set theory, proposed by Zadeh [1], has been extensively applied to many scientific fields. In fact, the field grew enormously, and applications were found in areas by many authors [2,3] as diverse as washing machines to handwriting recognition and other applications. Following the discovery of fuzzy sets, much attention has been paid to generalize the basic concepts of classical algebra in a fuzzy framework, and thus developing a theory of fuzzy algebras. In recent years, much interest is shown to generalize algebraic structures of groups, rings, modules, etc. The triangular norm, S-norm, originated from the studies of probabilistic metric spaces in which triangular inequalities were extended using the theory of S-norm. Later, Hohle [4], Alsina *et al.*, [5] introduced the S norm into fuzzy set theory and suggested that the S-norm be used for the intersection and union of fuzzy sets. Since then, many other researchers have presented various types of S-norms for particular purposes [6,7].

In practice, Zadeh's conventional S-norm,$\vee$, have been used in almost every design for fuzzy logic controllers and even in the modelling of other decision-making processes. However, some theoretical and experimental studies seem to indicate that other types of S-norms may work better in some situations, especially in the context of decision making processes. The author by using norms, investigated some properties of fuzzy algebraic structures [8–11].

The main purpose of the article is as follows: In Section 2, by using s-norms, we introduce anti fuzzy subgroups of group G. Next we prove that the union of them is also anti fuzzy subgroup and obtain some properties of them. Later, we define the composion between them and we prove that some results about them. Also we introduce anti fuzzy normal subgroups of group G under s-norms and we show that the union of any family of them is also anti fuzzy normal subgroup. Finally, we define normal subgroup between two anti fuzzy normal subgroups under s-norms and we investigate some basic properties of them. In Section 3, we investigate group bhomomorphisms and by using s-norms, we prove that image and pre image of anti fuzzy subgroups, anti fuzzy normal subgroups and normal subgroup between two anti fuzzy normal subgroups is also anti fuzzy subgroups, anti fuzzy normal subgroups and normal subgroup between two anti fuzzy normal subgroups, respectively. In Section 4, we define direct sum of anti fuzzy subgroups and anti fuzzy normal subgroups under s-norms and we prove that direct sum of any family of anti fuzzy subgroups and anti fuzzy normal subgroups under s-norms is also anti fuzzy subgroups and anti fuzzy normal subgroups under s-norms, respectively.

2. S-norms over anti fuzzy subgroups and anti fuzzy normal subgroups

Definition 1. [12] Let G be an arbitrary group with a multiplicative binary operation and identity e. A fuzzy subset of G, we mean a function from G into $[0,1]$. The set of all fuzzy subsets of G is called the $[0,1]$-power set of G and is denoted $[0,1]^G$.

Definition 2. [13] An s-norm S is a function $S:[0,1]\times[0,1]\to[0,1]$ having the following four properties:

(1) $S(x,0)=x$,
(2) $S(x,y)\leq S(x,z)$ if $y\leq z$,
(3) $S(x,y)=S(y,x)$,
(4) $S(x,S(y,z))=S(S(x,y),z)$,

for all $x,y,z\in[0,1]$.

We say that S is idempotent if for all $x\in[0,1]$, $S(x,x)=x$.

Example 1. The basic S-norms are

$$S_m(x,y)=\max\{x,y\},$$

$$S_b(x,y)=\min\{1,x+y\}$$

and

$$S_p(x,y)=x+y-xy$$

for all $x,y\in[0,1]$. (S_m is standard union, S_b is bounded sum, S_p is algebraic sum.)

Now we define anti fuzzy subgroup of G under an s-norm S.

Definition 3. Let μ be a fuzzy subset of a group G. Define μ is an anti fuzzy subgroup of G under an s-norm S iff

(1) $\mu(xy)\leq S(\mu(x),\mu(y))$,
(2) $\mu(x^{-1})\leq\mu(x)$,

for all $x,y\in G$.

Denote by $AFS(G)$, the set of all anti fuzzy subgroups of G under an s-norm S.

Example 2. Let $\mathbb{Z}$ be a set of integer and $G=(\mathbb{Z},+)$ be an additive group. Define $\mu:G\to[0,1]$ as

$$\mu_A(z)=\begin{cases}0.9 & \text{if } z\in 2\mathbb{Z};\\ 0.8 & \text{if } z\in\{2\mathbb{Z}-1\},\end{cases}$$

and let $S_p(x,y)=x+y-xy$ be an algebraic sum s-norm for all $x,y\in[0,1]$. Then $\mu\in AFS(G)$.

Proposition 1. *Let μ be a fuzzy subset of a finite group G and S be idempotent. μ satisfies condition (1) of Definition 3, then $\mu\in AFS(G)$.*

Proof. Let $x\in G, x\neq e$. Since G is finite, x has finite order, say $n>1$. So $x^n=e$ and $x^{-1}=x^{n-1}$. Now by using Definition 3(1) repeatedly, we have

$$\mu(x^{-1})=\mu(x^{n-1})=S(x^{n-2}x)\leq(\mu(x^{n-1}),\mu(x))\leq S(\underbrace{\mu(x),\mu(x),...,\mu(x)}_{n})=\mu(x).$$

Then $\mu \in AFS(G)$. □

In the following we define the union of two anti fuzzy subgroups of G under an s-norm S.

Definition 4. Let $\mu_1, \mu_2 \in AFS(G)$. We define

(1) $\mu_1 \subseteq \mu_2$ iff $\mu_1(x) \leq \mu_2(x)$,
(2) $\mu_1 = \mu_2$ iff $\mu_1(x) = \mu_2(x)$,
(3) $(\mu_1 \cup \mu_2)(x) = S(\mu_1(x), \mu_2(x))$,

for all $x \in G$.

Proposition 2. *Let $\mu_1, \mu_2 \in AFS(G)$, Then $\mu_1 \cup \mu_2 \in AFS(G)$.*

Proof. Let $x, y \in G$. Then

$$\begin{aligned}(\mu_1 \cup \mu_2)(xy) &= S(\mu_1(xy), \mu_2(xy)) \\ &\leq S(S(\mu_1(x), \mu_1(y)), S(\mu_2(x), \mu_2(y))) \\ &= S(S(\mu_1(x), \mu_2(x)), S(\mu_1(y), \mu_2(y))) \\ &= S((\mu_1 \cup \mu_2)(x), (\mu_1 \cup \mu_2)(y)).\end{aligned}$$

And

$$\begin{aligned}(\mu_1 \cup \mu_2)(x^{-1}) &= S(\mu_1(x^{-1}), \mu_2(x^{-1})) \\ &\leq S(\mu_1(x), \mu_2(x)) = (\mu_1 \cup \mu_2)(x).\end{aligned}$$

Thus $\mu_1 \cup \mu_2 \in AFS(G)$. □

Corollary 1. *Let $I_n = \{1, 2, ..., n\}$. If $\{\mu_i \mid i \in I_n\} \subseteq AFS(G)$, then $\mu = \cup_{i \in I_n} \mu_i \in AFS(G)$.*

Lemma 1. *Let $\mu \in AFS(G)$. If S be idempotent s-norm, then for all $x \in G$, and $n \geq 1$, we have*

(1) $\mu(e) \leq \mu(x)$,
(2) $\mu(x^n) \leq \mu(x)$,
(3) $\mu(x) = \mu(x^{-1})$.

Proof. Let $x \in G$ and $n \geq 1$. Then

(1) $\mu(e) = \mu(xx^{-1}) \leq S(\mu(x), \mu(x^{-1})) \leq S(\mu(x), \mu(x)) = \mu(x)$.
(2) $\mu(x^n) = \mu(\underbrace{xx...x}_{n}) \leq S(\underbrace{\mu(x), \mu(x), ..., \mu(x)}_{n}) = \mu(x)$.
(3) $\mu(x) = \mu((x^{-1}))^{-1} \leq \mu(x^{-1}) \leq \mu(x) = \mu(x^{-1})$ and so $\mu(x) = \mu(x^{-1})$.

□

Proposition 3. *Let $\mu \in AFS(G)$ and $x \in G$ and S be idempotent s-norm. Then $\mu(xy) = \mu(y)$ for all $y \in G$ if and only if $\mu(x) = \mu(e)$.*

Proof. Suppose that $\mu(xy) = \mu(y)$ for all $y \in G$. Then by letting $y = e$, we get that $\mu(x) = \mu(e)$. Conversely, suppose that $\mu(x) = \mu(e)$. By Lemma 1 we get that $\mu(x) \leq \mu(xy)$ and $\mu(x) \leq \mu(y)$. Now we have

$$\mu(xy) \leq S(\mu(x), \mu(y)) \leq S(\mu(y), \mu(y)) = \mu(y) = \mu(x^{-1}xy) \leq S(\mu(x), \mu(xy)) \leq S(\mu(xy), \mu(xy)) = \mu(xy).$$

Thus $\mu(xy) = \mu(y)$. □

Now we define the composition of two anti fuzzy subgroups of G under an s-norm S.

Definition 5. Let G be a set and let μ, ν be two fuzzy sets in G. Then $\mu o \nu$ is defined by

$$(\mu o \nu)(x) = \begin{cases} \inf_{x=ab} S((\mu(a), \nu(b)) & \text{if } x = ab, \\ 0 & \text{if } x \neq ab. \end{cases}$$

Proposition 4. *Let μ^{-1} be the inverse of μ such that $\mu^{-1}(x) = \mu(x^{-1})$. Then $\mu \in AFS(G)$ if and only if μ satisfies the following conditions:*

(1) $\mu o \mu \supseteq \mu$;
(2) $\mu^{-1} = \mu$.

Proof. Let $\mu \in AFS(G)$ and $x, y, z \in G$ such that $x = yz$. Then

$$\mu(x) = \mu(yz) \leq S(\mu(y), \mu(z)) = (\mu o \mu)(x),$$

so $\mu o \mu \supseteq \mu$. Also $\mu^{-1} = \mu$ comes from Lemma 1(3).

Conversely, let μ satisfies the condition (1) and (2). Then

$$\mu(yz) = \mu(x) \leq (\mu o \mu)(x) = \inf_{x=yz} S(\mu(y), \mu(z)) \leq S(\mu(y), \mu(z)).$$

Therefore $\mu \in AFS(G)$. □

Corollary 2. *Let $\mu, \nu \in AFS(G)$ and G be commutative group. Then $\mu o \nu \in AFS(G)$ if and only if $\mu o \nu = \nu o \mu$.*

Proof. If $\mu, \nu, \mu o \nu \in AFS(G)$, then from Proposition 3 we get that $\mu^{-1} = \mu, \nu^{-1} = \nu$ and $(\mu o \nu)^{-1} = (\mu o \nu)$. Then $\mu o \nu = \mu^{-1} o \nu^{-1} = (\nu o \mu)^{-1} = \nu o \mu$.

Conversely, since $\mu o \nu = \nu o \mu$ we have

$$(\mu o \nu)^{-1} = (\nu o \mu)^{-1} = \mu^{-1} o \nu^{-1} = \mu o \nu.$$

Also

$$(\mu o \nu) o (\mu o \nu) = \mu o (\nu o \mu) o \nu = \mu o (\mu o \nu) o \nu = (\mu o \mu) o (\nu o \nu) \subseteq \mu o \nu.$$

Now Proposition 3 gives us that $\mu o \nu \in AFS(G)$. □

In the following we define anti fuzzy normal subgroups of G under an s-norm S.

Definition 6. We say that $\mu \in AFS(G)$ is a normal if for all $x, y \in G$, $\mu(xyx^{-1}) = \mu(y)$. Also we denote by $AFNS(G)$ the set of all anti fuzzy normal subgroups of G under an s-norm S.

Proposition 5. *Let $\mu_1, \mu_2 \in AFNS(G)$. Then $\mu_1 \cup \mu_2 \in AFNS(G)$.*

Proof. Let $x, y, \in G$. Then

$$(\mu_1 \cup \mu_2)(xyx^{-1}) = S(\mu_1(xyx^{-1}), \mu_2(xyx^{-1})) = S(\mu_1(y), \mu_2(y)) = (\mu_1 \cup \mu_2)(y).$$

Therefore $\mu_1 \cup \mu_2 \in AFNS(G)$. □

Corollary 3. *Let $I_n = \{1, 2, ..., n\}$. If $\{\mu_i \mid i \in I_n\} \subseteq AFNS(G)$. Then $\mu = \cup_{i \in I_n} \mu_i \in AFNS(G)$.*

In the following we define the normal subgroups between two anti fuzzy subgroups of G under an s-norm S.

Definition 7. Let $\mu, \nu \in AFS(G)$ and $\mu \subseteq \nu$. Then μ is called a normal subgroup of the subgroup ν, written $\mu \trianglerighteq \nu$, if for all $x, y \in G$ we have that $\mu(xyx^{-1}) \leq S(\mu(y), \nu(x))$.

Proposition 6. *(1) If G_1 and G_2 are subgroups of G and G_1 is a normal subgroup of G_2, then $1_{G_1} \trianglerighteq 1_{G_2}$.*

(2) If S be idempotent s-norm, then every anti fuzzy subgroup under an s-norm S is an anti fuzzy normal subgroup of itself under an s-norm S.

Proof. (1) Let $x \in G_2$ and $y \in G_1$. If $G_1 \trianglerighteq G_2$, then $xyx^{-1} \in G_1$ and we have

$$1_{G_1}(xyx^{-1}) = 1 \leq 1 = S(1,1) = S(1_{G_1}(y), 1_{G_2}(x)).$$

Thus $1_{G_1} \trianglerighteq 1_{G_2}$.

(2) Let $\mu \in AFS(G)$ and $x, y \in G$. Then

$$\mu(xyx^{-1}) \leq S(\mu(xy), \mu(x^{-1})) \leq S(S(\mu(x), \mu(y)), \mu(x)) = S(S(\mu(x), \mu(x)), \mu(y)) = S(\mu(x), \mu(y)).$$

Therefore $\mu \trianglerighteq \mu$.

□

Proposition 7. *Let S be idempotent s-norm. If $\mu \in AFNS(G)$ and $\nu \in AFS(G)$, then $\mu \cup \nu \trianglerighteq \nu$.*

Proof. Proposition 1 gives us $(\mu \cup \nu) \in AFS(G)$. Now for all $x, y \in G$ we have

$$\begin{aligned}
(\mu \cup \nu)(xyx^{-1}) &= S(\mu(xyx^{-1}), \nu(xyx^{-1})) \\
&= S(\mu(y), \nu(xyx^{-1})) \\
&\leq S(\mu(y), S(\nu(xy)), \nu(x^{-1}))) \\
&\leq S(\mu(y), S(\nu(xy)), \nu(x))) \\
&\leq S(\mu(y), S(S(\nu(x), \nu(y)), \nu(x))) \\
&= S(\mu(y), S(S(\nu(x), \nu(x)), \nu(y))) \\
&= S(\mu(y), S(\nu(x), \nu(y))) \\
&= S(S(\mu(y), \nu(y)), \nu(x)) \\
&= S((\mu \cup \nu)(y), \nu(x)).
\end{aligned}$$

Hence $\mu \cup \nu \trianglerighteq \nu$. □

Lemma 2. *[14] Let S be an s-norm. Then*

$$S(S(x,y), S(w,z)) = S(S(x,w), S(y,z)),$$

for all $x, y, w, z \in [0,1]$.

Proposition 8. *Let S be idempotent s-norm and $\mu_1, \mu_2, \xi \in AFS(G)$. If $\mu_1, \mu_2 \trianglerighteq \xi$, then $\mu_1 \cup \mu_2 \trianglerighteq \xi$.*

Proof. By Proposition 2 we have that $\mu_1 \cup \mu_2 \in AFS(G)$. If $x, y \in G$, then

$$\begin{aligned}
(\mu_1 \cup \mu_2)(xyx^{-1}) &= S(\mu_1(xyx^{-1}), \mu_2(xyx^{-1})) \\
&\leq S(S(\mu_1(y), \xi(x)), S(\mu_2(y), \xi(x))) \\
&= S(S(\mu_1(y), \mu_2(y)), S(\xi(x), \xi(x))) \\
&= S(S(\mu_1(y), \mu_2(y)), \xi(x)) \\
&= S((\mu_1 \cup \mu_2)(y), \xi(x)).
\end{aligned}$$

Therefore, $\mu_1 \cup \mu_2 \trianglerighteq \xi$. □

Corollary 4. *Let $I_n = \{1, 2, ..., n\}$ and $\{\mu_i \mid i \in I_n\} \subseteq AFS(G)$ such that $\{\mu_i \mid i \in I_n\} \trianglerighteq \xi$. Then $\mu = \cup_{i \in I_n} \mu_i \trianglerighteq \xi$.*

3. Image and pre image of anti fuzzy subgroups and anti fuzzy normal subgroups w.r. s-norms under group homomorphisms

Definition 8. [7] Let f be a mapping from G into H, $\mu \in [0,1]^G$ and $\nu \in [0,1]^H$. Define $f(\mu) \in [0,1]^H$ and $f^{-1}(\nu) \in [0,1]^G$ as

$$f(\mu)(y) = \begin{cases} \inf\{\mu(x) \mid x \in G, f(x) = y\} & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{if } f^{-1}(y) = \emptyset. \end{cases}$$

and $f^{-1}(\nu)(x) = \nu(f(x))$ for all $x \in G$ and $y \in H$.

Proposition 9. *Let $\mu \in AFS(G)$ and H be a group. Suppose that f is a homomorphism of G into H. Then $f(\mu) \in AFS(H)$.*

Proof. Let $u, v \in H$ and $x, y \in G$ such that $u = f(x)$ and $v = f(y)$. Now

$$\begin{aligned} f(\mu)(uv) &= \inf\{\mu(xy) \mid u = f(x), v = f(y)\} \\ &\leq \inf\{S(\mu(x), \mu(y)) \mid u = f(x), v = f(y)\} \\ &= S(\inf\{\mu(x) \mid u = f(x)\}, \inf\{\mu(y) \mid v = f(y)\}) \\ &= S(f(\mu)(u), f(\mu)(v)). \end{aligned}$$

Also since $\mu \in AFS(G)$ we have

$$\begin{aligned} f(\mu)(u^{-1}) &= \inf\{\mu(x^{-1}) \mid x^{-1} \in G, f(x^{-1}) = u^{-1}\} \\ &\leq \inf\{\mu(x) \mid x \in G, f^{-1}(x) = u^{-1}\} \\ &= \inf\{\mu(x) \mid x \in G, f(x) = u\} = f(\mu)(u). \end{aligned}$$

Thus $f(\mu) \in AFS(H)$. □

Proposition 10. *Let H be a group and $\nu \in AFS(H)$. If f be a homomorphism of G into H, then $f^{-1}(\nu) \in AFS(G)$.*

Proof. Let $x, y \in G$. Then $f^{-1}(\nu)(xy) = \nu(f(xy)) = \nu(f(x)f(y) \leq S(\nu(f(x)), \nu(f(y))) = S(f^{-1}(\nu)(x), f^{-1}(\nu)(y))$. Also

$$f^{-1}(\nu)(x^{-1}) = \nu(f(x^{-1})) = \nu(f(x^{-1})) = \nu(f^{-1}(x)) \leq \nu(f(x)) = f^{-1}(\nu)(x).$$

Thus $f^{-1}(\nu) \in AFS(G)$. □

Proposition 11. *Let $\mu \in AFNS(G)$ and H be a group. Suppose that f is an epimorphism of G onto H. Then $f(\mu) \in AFNS(H)$.*

Proof. From Proposition 10 we have $f(\mu) \in AFS(H)$. Let $x, y \in H$. Since f is a surjection, $f(u) = x$ for some $u \in G$. Then

$$\begin{aligned} f(\mu)(xyx^{-1}) &= \inf\{\mu(w) \mid w \in G, f(w) = xyx^{-1}\} \\ &= \inf\{\mu(u^{-1}wu) \mid w \in G, f(u^{-1}wu) = y\} = \inf\{\mu(w) \mid w \in G, f(w) = y\} \\ &= f(\mu)(y). \end{aligned}$$

Then $f(\mu) \in AFNS(H)$. □

Proposition 12. *Let H be a group and $\nu \in AFNS(H)$. Suppose that f is a homomorphism of G into H. Then $f^{-1}(\nu) \in AFNS(G)$.*

Proof. By Proposition 11 we obtain that $f^{-1}(\nu) \in AFS(G)$. Now for any $x, y \in G$, we have

$$f^{-1}(\nu)(xyx^{-1}) = \nu(f(xyx^{-1})) = \nu(f(x)f(y)f(x^{-1})) = \nu(f(x)f(y)f^{-1}(x)) = \nu(f(y)) = f^{-1}(\nu)(y).$$

Hence $f^{-1}(\nu) \in AFNS(G)$. □

Proposition 13. *Let $\mu, \nu \in AFS(G)$ and $\mu \trianglerighteq \nu$. Let H be a group and f a homomorphism from G into H. Then $f(\mu) \trianglerighteq f(\nu)$.*

Proof. As Proposition 10 we have $f(\mu), f(\nu) \in AFS(H)$. Let $x, y \in H$ and $u, v \in G$. Then,

$$\begin{aligned} f(\mu)(xyx^{-1}) &= \inf\{\mu(z) \mid z \in G, f(z) = xyx^{-1}\} \\ &= \inf\{\mu(uvu^{-1}) \mid u, v \in G, f(u) = x, f(v) = y\} \\ &\leq \inf\{S(\mu(v), \nu(u)) \mid f(u) = x, f(v) = y\} \\ &= S(\inf\{\mu(v) \mid y = f(v)\}, \inf\{\nu(u) \mid x = f(u)\}) \\ &= S(f(\mu)(y), f(\nu)(x)). \end{aligned}$$

Hence $f(\mu) \trianglerighteq f(\nu)$. □

Proposition 14. *Let H be a group. Let $\mu, \nu \in AFS(H)$ and $\mu \trianglerighteq \nu$. If f be a homomorphism from G into H, then $f^{-1}(\mu) \trianglerighteq f^{-1}(\nu)$.*

Proof. Using Proposition 11 we have $f^{-1}(\mu), f^{-1}(\nu) \in AFS(G)$. Let $x, y \in G$. Now

$$f^{-1}(\mu)(xyx^{-1}) = \mu(f(xyx^{-1})) = \mu(f(x)f(y)f^{-1}(x)) \leq S(\mu(f(y)), \nu(f(x))) = S(f^{-1}(\mu)(y), f^{-1}(\nu)(x)).$$

Hence $f^{-1}(\mu) \trianglerighteq f^{-1}(\nu)$. □

4. Direct sum of anti fuzzy subgroups and anti fuzzy normal subgroups under s-norms

Definition 9. Let μ and ν be anti fuzzy subgroups of the groups G and H, respectively under an s-norm S. The direct sum of μ and ν, denoted by $\mu \oplus \nu$, is the function defined by setting for all x in G and y in H,$(\mu \oplus \nu)(x, y) = S(\mu(x), \nu(y))$.

Proposition 15. *Let $\mu_i \in AFS(G_i)$ for $i = 1, 2$. Then $\mu_1 \oplus \mu_2 \in AFS(G_1 \oplus G_2)$.*

Proof. Let $(a_1, b_1), (a_2, b_2) \in G_1 \oplus G_2$. Then

$$\begin{aligned} (\mu_1 \oplus \mu_2)((a_1, b_1)(a_2, b_2)) &= (\mu_1 \oplus \mu_2)(a_1a_2, b_1b_2) \\ &= S(\mu_1(a_1a_2), \mu_2(b_1b_2)) \\ &\leq S(S(\mu_1(a_1), \mu_1(a_2)), S(\mu_2(b_1), \mu_2(b_2))) \\ &= S(S(\mu_1(a_1), \mu_2(b_1), S(\mu_1(a_2), \mu_2(b_2)) \\ &= S((\mu_1 \oplus \mu_2)(a_1, b_1), (\mu_1 \oplus \mu_2)(a_2, b_2)). \end{aligned}$$

Also

$$(\mu_1 \oplus \mu_2)(a_1, b_1)^{-1} = (\mu_1 \oplus \mu_2)(a_1^{-1}, b_1^{-1}) = S(\mu_1(a_1^{-1}), \mu_2(b_1^{-1})) \leq S(\mu_1(a_1), \mu_2(b_1)).$$

Hence $\mu_1 \oplus \mu_2 \in AFS(G_1 \oplus G_2)$. □

Corollary 5. *Let $\mu \in AFS(G)$ and $\nu \in TF(H)$. Then $\mu \oplus 1_H, 1_G \oplus \nu \in AFS(G \oplus H)$.*

Corollary 6. *Let $\mu_i \in AFS(G_i)$ for $i = 1, 2, ..., n$. Then $\mu_1 \oplus \mu_2 \oplus ... \oplus \mu_n \in AFS(G_1 \oplus G_2 \oplus ... \oplus G_n)$.*

Proposition 16. *Let $\mu_i \in AFNS(G_i)$ for $i = 1, 2$. Then $\mu_1 \oplus \mu_2 \in AFNS(G_1 \oplus G_2)$.*

Proof. Let $(a_1,b_1),(a_2,b_2) \in G_1 \oplus G_2$. Then

$$\begin{aligned}(\mu_1 \oplus \mu_2)((a_1,b_1)(a_2,b_2)(a_1,b_1)^{-1}) &= (\mu_1 \oplus \mu_2)((a_1,b_1)(a_2,b_2)(a_1^{-1},b_1^{-1}))\\ &= (\mu_1 \oplus \mu_2)(a_1a_2a_1^{-1},b_1b_2b_1^{-1})\\ &= S(\mu_1(a_1a_2a_1^{-1}),\mu_2(b_1b_2b_1^{-1}))\\ &= S(\mu_1(a_2),\mu_2(b_2))\\ &= (\mu_1 \oplus \mu_2)(a_2,b_2).\end{aligned}$$

Therefore $\mu_1 \oplus \mu_2 \in AFNS(G_1 \oplus G_2)$. □

Corollary 7. *Let $\mu_i \in AFNS(G_i)$ for $i = 1,2,...,n$. Then $\mu_1 \oplus \mu_2 \oplus ... \oplus \mu_n \in AFNS(G_1 \oplus G_2 \oplus ... \oplus G_n)$.*

Proposition 17. *Let $\mu_i, \nu_i \in AFS(G_i)$ and $\mu_i \subseteq \nu_i$ for $i = 1,2$. If $\mu_i \trianglerighteq \nu_i$, then $\mu_1 \oplus \mu_2 \trianglerighteq \nu_1 \oplus \nu_2$.*

Proof. Let $(a_1,b_1),(a_2,b_2) \in G_1 \oplus G_2$. Then

$$\begin{aligned}(\mu_1 \oplus \mu_2)((a_1,b_1)(a_2,b_2)(a_1,b_1)^{-1}) &= (\mu_1 \oplus \mu_2)((a_1,b_1)(a_2,b_2)(a_1^{-1},b_1^{-1}))\\ &= (\mu_1 \oplus \mu_2)(a_1a_2a_1^{-1},b_1b_2b_1^{-1})\\ &= S(\mu_1(a_1a_2a_1^{-1}),\mu_2(b_1b_2b_1^{-1}))\\ &\le S(S(\mu_1(a_2),\nu_1(a_1)),S(\mu_2(b_2),\nu_2(b_1)))\\ &= S(S(\mu_1(a_2),\mu_2(b_2)),S(\nu_1(a_1),\nu_2(b_1))) =\\ &S((\mu_1 \oplus \mu_2)(a_2,b_2),(\nu_1 \oplus \nu_2)(a_1,b_1)).\end{aligned}$$

Thus $\mu_1 \oplus \mu_2 \trianglerighteq \nu_1 \oplus \nu_2$. □

Corollary 8. *Let $\mu_i \trianglerighteq \nu_i$ for $i = 1,2,...,n$. Then $\mu_1 \oplus \mu_2 \oplus ... \oplus \mu_n \trianglerighteq \nu_1 \oplus \nu_2 \oplus ... \oplus \nu_n$.*

5. Conclusion

As using S-norms, anti fuzzy subgroups and anti fuzzy normal subgroups were defined which are new notions and were considered their fundamental properties and also was made an attempt to study the characterizations of them. Next image and pre image of them were investigated under group homomorphisms. Finally, the direct sum of them was introduced and was proved that direct sum of any familly of them be also anti fuzzy subgroups and anti fuzzy normal subgroups under S-norms, respectively.

Acknowledgments: The author would like to thank the referees for carefully reading the manuscript and making several helpful comments to increase the quality of the paper.

Conflicts of Interest: "The author declares no conflict of interest."

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